Basic terms

Sample space is the collection of all possible outcomes.
Events is a specific collection of outcomes.
Mutually exclusive indicates that the events are disjoint.
Collectively exhaustive indicates that the events cover the entire sample space.

n-nomial Expansions, Theorems and Identities

$(a+b)^n$ = Sum over k $({}^nC_k a^k b^{(n-k)})$ where k goes from 0 to n
Binomial coeff. identity: ${}^nC_k = {}^{(n-1)}C_{(k-1)} + {}^{(n-1)}C_k$ where first term corresponds to A and second to A^C
${}^nC_m = {}^nC_{n-m}$
Sum over r $({}^nC_r(-1)^r(1)^{(n+r)}$ is 0 where r goes from 0 to n
Sum over r $(({}^nC_r)^2) = {}^{2n}C_n$ where r goes from 0 to n
Sum over s $({}^sC_m)$ = ^(n+1)C~(m+1) where s goes from m to n
Hypergeometric expansion: ${}^{(n+m)}C_r = {}^nC_0{}^mC_r + {}^nC_1{}^mC_{(r-1)} + ... + {}^nC_r{}^mC_0$ which is a CE and ME enumeration
$n! = (n/e)^n$ x root(2n x pi) - Stirling's approx. for n!

n-nomial Expansions, Theorems and Identities (cont)

Trinomial expansion: $(a+b+c)^n$ = sum over i, j, k $(C'a^ib^jc^k)$ where i, j, k go from 0 to n and i+j+k=n and C'=n!/(i!j!k!)
In an n-nomial expansion $(a_1 + ... + a_r)^n$, the # of terms in the sum is ${}^r\overline{C}_n = {}^{(r+n-1)}C_n = {}^{(r+n-1)}C_{(r-1)}$

Basics of set theory

S=entire sample space=A u A^C where A = any subset of S
Null set (contains 0 members) = phi
S^C=phi and phi^C=S
$(A^C)^C$=A
A n B = A.B = AB for all x iff x in A and x in B
A u B = {x|x in A and/or x in B}
A - B = A n B^C
If A and B are ME, A n B = phi
If events E_1 to E_n are CE, then E_1 u ... u E_n = S
Commutative law: A u B = B u A and A n B = B n A
Associative law : A u (B u C) = (A u B) u C, similarly for intersection
Distributive laws: A u (B n C) = (A u B) n (A u C) and A n (B u C) = (A n B) u (A n C)
A n phi = phi and A u phi = A

Basics of set theory (cont)

A n S = A = A n (B u B^C)
A u B = A u (A^C n B) = B u (B^C n A)
A = B iff A is a subset of B and vice versa

Formulations of Probability

0<=P(E)<=1
P(S)=1 where S is the entire sample space
For ME events A and B, P(A u B) = P(A) + P(B)
1 = P(S) = P(E u E^C) = P(E) + P(E^C)
Probabilities for ME events are equal
P(A n B) = P(A)P(B)
Inclusion-exclusion principle: P(E_1 u ... u E_n) = (sum(P(E_i)) for i=1..n) - (sum(P(E_iE_{i+1})) for i=1...n-1) +... +/- P($E_1E_2...E_n$), ie: singles - pairs + triples - quadruples + ... +/- n-tuple

Counting Principles

Sum rule for or cases and product rule for and cases
Permutations consider order, combinations do not
Permutations of n distinct objs. take k = ${}^nP_k$ = (n!)/(n-k)!
Permutations of n distinct objs. take n w/ r groups of indistinct objs. = (n!)/(n_1! ... n_r!)

Counting Principles (cont)

Combinations of n objs. take k = ${}^nC_k$ = (n!)/((n-k)! * k!)
${}^n\overline{P}_r = n^r$ and ${}^n\overline{C}_r = {}^{(n+r-1)}C_r$: used for situations involving permutations and combinations of n objs. with replacement taken k at a time

Conditional Probability

P(A|B) = P(AB)/P(B)
Bayes' Rule: P(A|B)=(P(B|A)P(A))/P(B)=(P(B|A)P(A))/(P(B|A)P(A) + P(B|A^C)P(A^C))
Conditional probability is not symmetric

Hat Matching Problem

* n men throw their hats on the floor and each man randomly picks up his hat.
* If k men out of n draw their own hats, then the remaining men don't draw their own hats. Hence, the probability of k men drawing their own hats (over all k-tuples) = $({}^nC_k(n-k)!)/n!$ = 1/k!
* P(k matches) = calculate the intersection probability using inclusion-exclusion principle and divide by k!

Problem Patterns, Tips and Solutions

*** Always sketch trees for problems involving product rule, with smaller use cases**

*** 5-card draw poker:**

If the number of cards is n (usually 52), total # of ways of drawing 5 cards is ${}^{n}C_{5}$.

Number of ways to pick one card value = ${}^{13}C_{1}$

Number of ways to pick k suits = ${}^{4}C_{k}$ where k can go from 1 to 4.

*** Balls and bins:**

If there are n balls and r bins, there can be 2 scenarios:

- if the balls are distinguishable, each ball can go into any one of r bins. Thus the # of distinct permutations would be ${}^{r}P_{n} = r^{n}$
- if the balls are indistinguishable, there will be 2 cases:

* No bin should be empty. Occupancy vector is $x_1+...+x_r=n$ where every x is at least 1. In this case, there can be n-1 possible locations for bin separators from which we can choose r-1 to ensure at least 1 ball in each bin. # of possible arrangements = ${}^{(n-1)}C_{(r-1)}$.

Problem Patterns, Tips and Solutions (cont)

* A bin can have no balls. Then the occupancy vector would be $y_1+...+y_r=n+r$ and the # of arrangements will be ${}^{(n+r-1)}C_{(r-1)}$

***Selecting committees:**

Can be solved using product rule or hypergeometric approach.

***Drawing the only special ball from n balls in k trials:**

Total outcomes = ${}^{n}C_{k} = {}^{(1+(n-1))}C_{k} = {}^{1}C_{0}\,{}^{(n-1)}C_{k} + {}^{1}C_{1}\,{}^{(n-1)}C_{(k-1)}$ - the first term is for no special ball and the second term is for the special ball. Alternatively, we can calculate using a tree diagram.

***Bridge hands:**

Remember: each hand has 13 cards.

***Roundtable arrangements:**

If there are k people sitting at a roundtable, total # of arrangements is k!/k