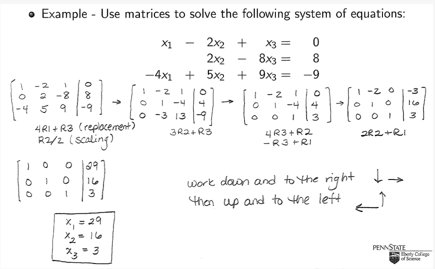
1.1

A Matrix	row, columns
Coefficients Matrix	Just Left Hand Side
Augmented Matrix	Left and Right Hand Side
Solving Linear Systems	(1) Augmented Matrix (2) Row Operations (3) Solution to Linear System The RHS is the solution
One Solution	Upper triangle with Augmented Matrix
No Solution	Last row is all zeros = RHS number
Infinitely Many Solutions	Last row (including RHS) is all zeros
Inconsistent	Has No Solution

1.1 Example(1)



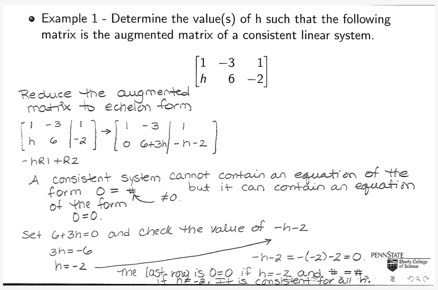
1.2

Echelon Matrix	(1) Zero Rows at the bottom (2) Leading Entries are down and to the right (3) Zeros are below each leading entry

1.2 (cont)

Reduced Echelon Matrix	(1) The leading entry of each nonzero row is 1 (2) Zeros are below AND above each 1
Pivot Position	Location of Matrix that Corresponds to a leading 1 in REF
Pivot Column	Column in Matrix that contains a pivot
To get to EF	down and right
To get to REF	up and left
Free Variables	Variables that don't correspond to pivot columns
Consistent System	Pivot in every Column

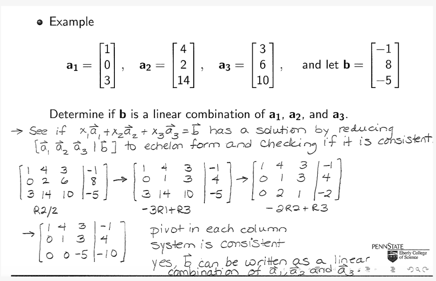
1.2 Example (1)



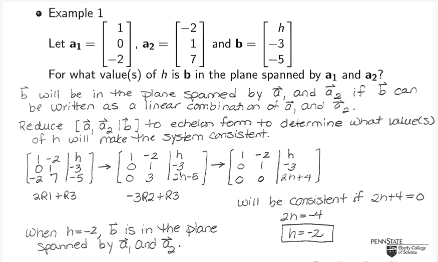
1.3

RR^2	Set of all vectors with 2 rows

1.3 Example (1)



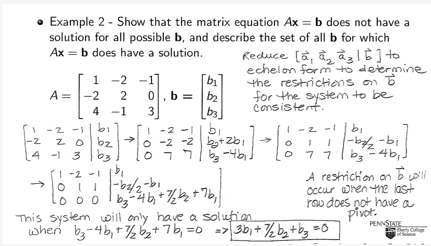
1.3 Example (2)



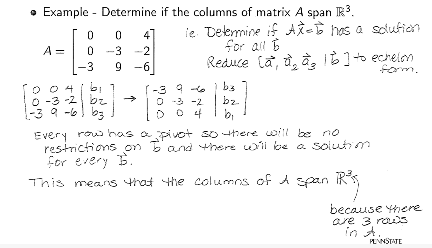
1.4

Vector Equation	x1**a1**+x2**a2**+x3**a3** =**b**
Matrix Equation	A**x**=**b**
If A is an m x n matrix the following are all true or all false	A**x** = **b** has a solution for every **b** in RR^m Every **b** in RR^m is a lin. combo of columns in A Columns of A span RR^m Matrix A has a pivot in every row (i.e. no row of zeros)
Anything in **Bold** means it is a vector.	

1.4 Example (1)



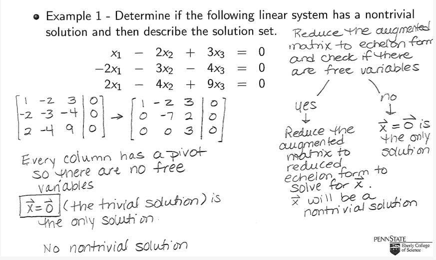
1.4 Example (2)



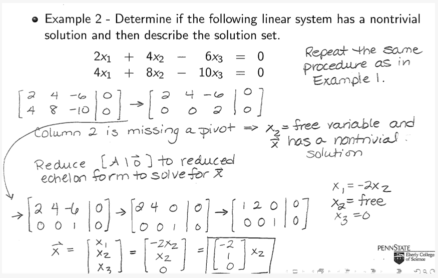
1.5

Homogeneous	A**x** = **0**
Trivial Solution	A**x** = **0** if at lease one column is missing a pivot
Determine if homogenous Linear System has a non trivial solution	(1) Write as Augmented Matrix (2) Reduce to EF (3) Determine if there are any free variables- (column w/o pivot) (4) If any free variables, than a non-trivial solution exists (5) Non-Trivial Solution can be found by further reducing to REF and solving for **x**
If A**x** = **0** has one free variable	Than **x** is a line that passes through the origin
If A**x** = **0** has two free variables	Than **x** has a plane that passes through the origin

1.5 Example (1)



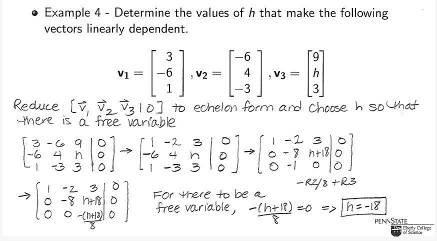
1.5 Example (2)



1.7

Linear Independence	No free Variables, none of the vectors are multiples of each other
To check ind/dep	reduce augmented matrix to EF and see if there are free variables(ie. every column must have a pivot to be linearly independent)
To check if multiples	**u** = c * **v** find value of c, then it is a multiple therefore linearly dependent
Linearly Dependent	If there are more columns than rows

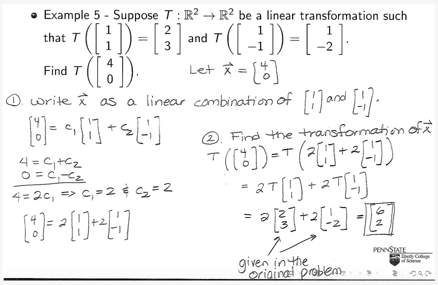
1.7 Example (1)



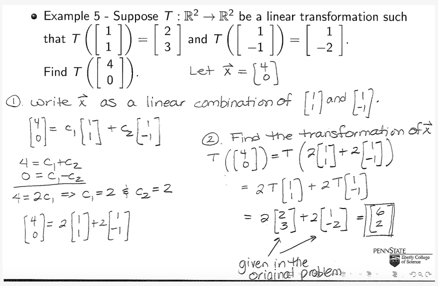
1.8

Every Matrix Transformation is a:	Linear Transformation
T(**x**) =	A(**x**)
If A is m x n Matrix, then the properties are	(1) T(**u** + **v**) = T(**u**) + T(**v**) (2) T(c**u**) = cT(**u**) (3) T(**0**) = **0** (4) T(c**u** + d**v**) = cT(**u**) + dT(**v**)

1.8 Example (1)



1.8 Example (2)



1.9

RR^n --> RR^m is said to be 'onto'	Equation T(**x**) =A**x**=**b** has a unique solution or more than one solution each row has a pivot
RR^n --> RR^m is said to be one-to-one	Equation T(**x**) =A**x**=**b** has a unique solution or no solution each row has a pivot

2.1

Addition of Matrices	Can Add matrices if they have same # of rows and columns (ie **A**(3x4) and **B**(3x4) so you can add them)
Multiply by Scalar	Multiply each entry by scalar
Matrix Multiplication (**A** x **B**)	Must each row of **A** by each column of **B**
Powers of a Matrix	Can compute powers by if the matrix has the same number of columns as rows
Transpose of Matrix	row 1 of A becomes column 1 of A row 2 of A becomes column 2 of A

2.1 (cont)

Properties of Transpose	(1) if **A** is m x n, then $\mathbf{A}^T$ is n x m (2) $(\mathbf{A}^T)^T = \mathbf{A}$ (3) $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$ (4) $(t\mathbf{A})^T = t\mathbf{A}^T$ (5) $(\mathbf{A}\,\mathbf{B})^T = \mathbf{B}^T\,\mathbf{A}^T$

2.2

Singular matrix	A matrix that is NOT investable
Determinate of A (2 x 2) Matrix	det A = ad - bc
If A is invertable & (nxn)	There will never be no solution or infinitely many solutions to A**x** = **b**
Properties of Invertable Matricies	$(\mathbf{A}^{-1})^{-1} = \mathbf{A}$ (assuming **A** & **B** are investable) $(AB)^{-1} = B^{-1} A^{-1}$ $(A^T)^{-1} = (A^{-1})^T$
Finding Inverse Matrix	[A | I] --> [I | A^{-1}] *Use row operations* STOP when you get a row of Zeros, it cannot be reduced

2.2 Example (1)

- Example 2 - Let A, B, C and X be $n \times n$ invertible matrices. Solve $B(X+A)^{-1} = C$ for the matrix X.

Method 1

Get $(X+A)^{-1}$ by itself. 1st.

$B(X+A)^{-1} = C$

$B^{-1}B(X+A)^{-1} = B^{-1}C$

Note: If you multiply on the left for one side, you must multiply on the left on the other side

$I(X+A)^{-1} = B^{-1}C$

$(X+A)^{-1} = B^{-1}C$

$((X+A)^{-1})^{-1} = (B^{-1}C)^{-1}$

$X+A = C^{-1}(B^{-1})^{-1}$

$\boxed{X = C^{-1}B - A}$

Method 2

$B(X+A)^{-1} = C$

$(B(X+A)^{-1})^{-1} = C^{-1}$

$((X+A)^{-1})^{-1}B^{-1} = C^{-1}$

$(X+A)(B^{-1}) = C^{-1}$

$XB^{-1} + AB^{-1} = C^{-1}$

$XB^{-1} = C^{-1} - AB^{-1}$

$XB^{-1}B = (C^{-1} - AB^{-1})B$

$X = C^{-1}B - AB^{-1}B$

$\boxed{X = C^{-1}B - A}$

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2.3 Invertable Matrix Theorem

- The Invertible Matrix Theorem - Let A be a square $n \times n$ matrix. Then all of the following statements are equivalent:
  (a) A is an invertible matrix
  (b) A is row equivalent to the $n \times n$ identity matrix I.
  (c) A has n pivots.
  (d) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
  (e) The columns of A form a linearly independent set.
  (f) The linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one.
  (g) The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
  (h) The columns of A span $\mathbb{R}^n$.
  (i) The linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is onto.
  (j) There is an $n \times n$ matrix C such that $CA = I$.
  (k) There is an $n \times n$ matrix D such that $AD = I$.
  (l) A^T is invertible.

The above theorem states that if one of these is false, they all must be false. If one is true, then they are all true.

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2.8

A subspace *S* of RR^n is a subspace is *S* satisfies:	(1) *S* contains zero vector (2) If **u** & **v** are in *S*, then **u** + **v** is also in *S* (3) If *r* is a real # & **u** is in *S*, then *r***u** is also in *S*
Subspace RR^3	Any Plane that Passes through the origin forms a subspace RR^3 Any set that contains nonlinear terms will NOT form a subspace RR^3
Null Space (Nul A)	To determine in **u** is in the Nul(A), check if: A**u** = **0** If yes --> then **u** is in the Nullspace

2.8 Example (1)

- Example 1 - Given the following matrix A and an echelon form of A, find a basis for Col A.

$$A = \begin{bmatrix} 3 & -6 & 9 & 0 \\ 2 & -4 & 7 & 2 \\ 3 & -6 & 6 & -6 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & -2 & 5 & 4 \\ 0 & 0 & \boxed{3} & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

echelon form of A.

pivot columns = columns 1 and 3.

basis for Col A = pivot columns of A (not pivot columns of echelon form of A)

basis for Col A = $\left\{ \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 9 \\ 7 \\ 6 \end{bmatrix} \right\}$

Col A = $c_1 \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 9 \\ 7 \\ 6 \end{bmatrix}$

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2.8 Example (2)

- Example 2 - Given the following matrix A and an echelon form of A, find a basis for Nul A.

$$A = \begin{bmatrix} 3 & -6 & 9 & 0 \\ 2 & -4 & 7 & 2 \\ 3 & -6 & 6 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 & 4 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduce $[A \mid \vec{0}]$ to reduced echelon form to solve for $\vec{x}$

$$\left[\begin{array}{cccc|c} 1 & -2 & 5 & 4 & 0 \\ 0 & 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -2 & 5 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -2 & 0 & -6 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = -x_4$
$x_1 = 2x_2 + 6x_4$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_2 + 6x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 6 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$\boxed{\text{Basis for Nul } A = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}}$

vectors in basis for Nul A = # free variables.

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2.9

Dimension of a non-zero Subspace	# of vectors in any basis; it is the # of linearly independent vectors
Dimension of a zero Subspace	is Zero
Dimension of a Column Space	# of pivot columns
Dimension of a Null Space	# of free variables in the solution A**x**=**0**
Rank of a Matrix	# of pivot columns
The Rank Theorem	Matrix A has *n* columns: rank A (# pivots) + dim Nul A (# free var.) = *n*

dim = dimension; var. = variable

2.9 Refrence

- The Invertible Matrix Theorem Continued- Let A be a square $n \times n$ matrix. Then all of the following statements are equivalent to the statement that A is an invertible matrix. (see Section 2.3)
  - (m) The columns of A form a basis of $\mathbb{R}^n$ (because they are linearly independent)
  - (n) Col $A = \mathbb{R}^n$
  - (o) dim Col $A = n$ (because n pivots)
  - (p) rank $A = n$ (because n pivots)
  - (q) Nul $A = \{\mathbf{0}\}$ (because no free variables)
  - (r) dim Nul $A = 0$ (because no free variables).

Because every vector in $\mathbb{R}^n$ can be written as a linear combination of the columns of A.

$\underbrace{\text{rank } A}_{n} + \underbrace{\text{dim Nul } A}_{0} = n$

3.1

Calculating Determinant of Matrix A is another way to tell if a linear system of equations has a solution	(1) Det(A) not =0, then A**x**=**b** has a unique solution (2) Det(A) =0, then A**x**=**b** has no solutions or inf many
If A**x** not= **0**	A^{-1} exist
If A**x** = **0**	A^{-1} Does NOT exist
Cofactor Expansion	Use row/column w/ most zeros
If Matrix A has an upper or lower triangle of zeros	The det(A) is the multiplication down the diagonals

3.1 Reference (1)

2. Determinate of a 3 × 3 Matrix

- Let A be an $n \times n$ matrix (or here a 3x3 matrix)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- The determinate of matrix A is given as follows:

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$$

notice the alternating sign.

$$\det A_{11} = a_{22}a_{33} - a_{23}a_{32} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$
$$\det A_{12} = a_{21}a_{33} - a_{23}a_{31} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$
$$\det A_{13} = a_{21}a_{32} - a_{22}a_{31} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

- This formula utilizes a cofactor expansion across the first row

3.1 Example (1)

- Example - Compute the determinate of

$$A = \begin{bmatrix} -3 & 1 & 2 \\ 5 & 5 & -8 \\ 4 & 2 & -5 \end{bmatrix}$$

$$\begin{array}{ccc} + & - & + \end{array}$$
$$\det A = \begin{vmatrix} -3 & 1 & 2 \\ 5 & 5 & -8 \\ 4 & 2 & -5 \end{vmatrix} = +(-3)\begin{vmatrix} 5 & -8 \\ 2 & -5 \end{vmatrix} - 1 \begin{vmatrix} 5 & -8 \\ 4 & -5 \end{vmatrix} + 2 \begin{vmatrix} 5 & 5 \\ 4 & 2 \end{vmatrix}$$
$$= -3(-25+16) - 1(-25+32) + 2(10-20)$$
$$= -3(-9) - 1(7) + 2(-10)$$
$$= 27 - 7 - 20$$
$$= \boxed{0}$$

← this means A^{-1} does not exist and $A\vec{x} = \vec{b}$ has no solution or infinitely many solutions.

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3.1 Reference (2)

- Shortcut Method for a 3 × 3 matrix

add column 1 & 2 to the right of the matrix

do not memorize this. (learn the pattern instead)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$
$$\begin{array}{ccccc} - & - & + & + & + \end{array}$$

$$\det = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31}$$

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3.1 Example (2)

- Example 2 - Use a cofactor expansion to compute the determinate of

$$A = \begin{array}{c} \begin{array}{cccc} + & - & + & - \end{array} \\ \begin{bmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{bmatrix} \end{array}$$

Choose the row or column with the most zeros => Choose row 2.

$$\det A = -3 \begin{vmatrix} 1 & -2 & 2 \\ 2 & -6 & 5 \\ 5 & 0 & 4 \end{vmatrix} = -3 \left[+5 \begin{vmatrix} -2 & 2 \\ -6 & 5 \end{vmatrix} - 0 \begin{vmatrix} 1 & 2 \\ -2 & 5 \end{vmatrix} + 4 \begin{vmatrix} 1 & -2 \\ 2 & -6 \end{vmatrix} \right]$$
$$= -3 \left[5(-10+12) - 0 + 4(-6+4) \right]$$
$$= -3(10-8) = \boxed{-6}$$

You can also use the shortcut 3X3 method to evaluate this.

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3.2

Determinate Property 1	If a multiple of 1 row of A is added to another row to produce Matrix B, then det(B)=det(A)
Determinate Property 2	If 2 rows of A are interchanged to produce B, then det(B)=-det(A)
Determinate Property 3	If one row of A is multiplied to produce B, then det(B)=k*det(A)

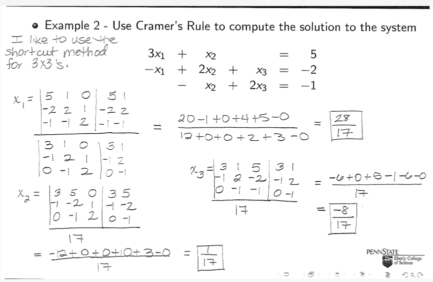
3.2 (cont)

Assuming both A & B are n x n Matrices	(1) $\det(A^T) = \det(A)$ (2) det(AB) = det(A)*det(B) (3) $\det(A^{-1}) = 1/\det(A)$ (4) $\det(cA) = c^n \det(A)$ (5) $\det(A^r) = (\det A)^r$

3.3 AKA Cramer's Rule

Cramer's Rule	Can be used to find the solution to a linear system of equations A**x**=**b** when A is an investable square matrix
Def. of Cramer's Rule	Let A be an n x n invertible matrix. For any **b** in RR^n, the unique solution **x** of A**x**=**b** has entries given by xi = detAi(**b**)/det(A) *i = 1,2,...n*
Ai(**b**)	is the matrix A w/ column i replaced w/ vector **b**

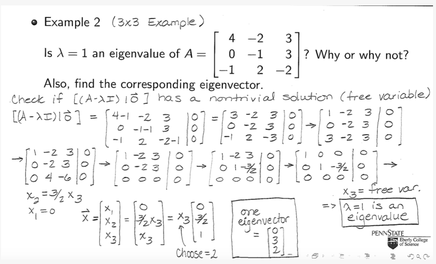
3.3 Example (1)



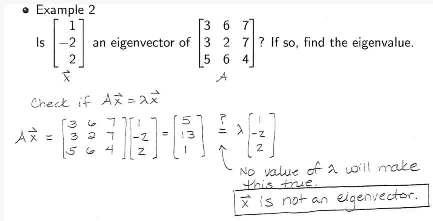
5.1

A**u**=λ**u**	A is an nxn matrix. A nonzero vector **u** is an eigenvector of A if there exists such a scalar λ
To determine if λ is an eigenvalue	reduce [(A-λI)|0] to echelon form and see if it has any free variables. yes -> λ is Eigenvalue no -> λ is not eigenvalue
To determine if given vector is an eigenvector	Ax=λ**x**
Eigenspace of A =	Nullspace of (A-λI)
Eigenvalues of triangular Matrix	entries along diagonal *you CANNOT row reduce a matrix to find its eigenvalues

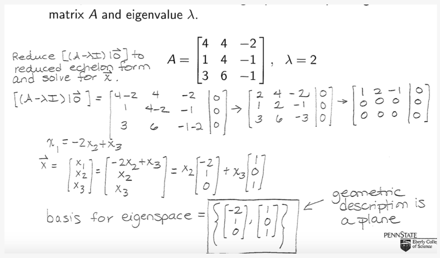
5.1 Example (1)



5.1 Example (2)



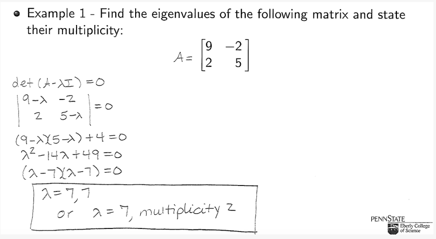
5.1 Example (3)



5.2

If λ is an eigenvalue of a Matrix A	then (A-λI)**x**=**0** will have a nontrivial solution
A nontrivial solution will exist	if det(A-λI)=0 (Characteristic Equation)
A is nxn Matrix. A is invertible if and only if	(1) The # 0 is NOT an λ of A (2) The det(A) is not zero
Similar Matrices	If nxn Matrices A and B are similar, then they have the same characteristic polynomial (same λ) with same multiplicities

5.2 Example (1)



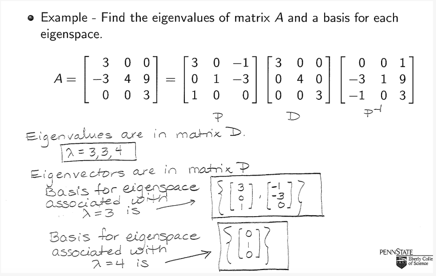
5.3

A matrix A written in diagonal form	$A=PDP^{-1}$
Power of Matrix	A^k = Diagonal matrix and #'s on diagonal get raised to the k
Determining if Matrix is Diagonalizable	
λ of a nxn matrix	n distinct (or real) λ then matrix is diagonalizable less than n λ, it may or may not be diagonalizable; it will be if # of linearly dependent eigenvectors = n
eigenvectors of nxn matrix	n linearly independent eigenvectors, then diagonalizable less than n linearly independent eigenvectors, then matrix is NOT diagonlizable
D	matrix w/ λ down diagonal

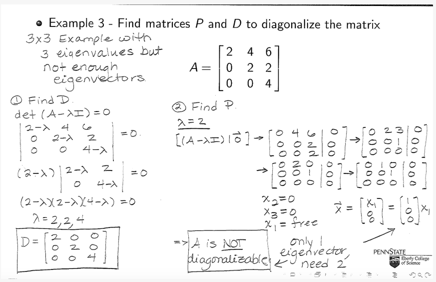
5.3 (cont)

P	columns of P have linearly n linearly independent eigenvectors
Finding *P*	solve A-λI and plug in the λ values. Reduce to EF, solve for **x**, & find eigenvector

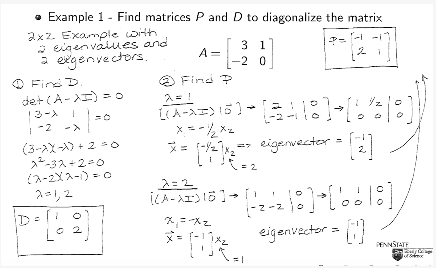
5.3 Example (1)



5.3 Example (2)



5.3 Example (3)



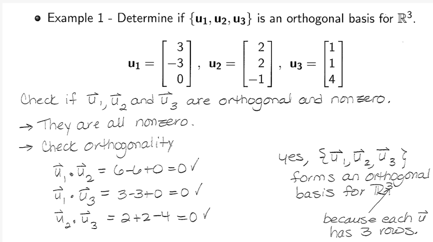
6.1

Length of vector **x**	$\|\|$**x**$\|\|$ = sqrt(x1^2+x2^2)
Length fo vector **x** in RR^2	$\|\|$**x**$\|\|$ = sqrt(**x** • **x**)
The Unit Vector	**u** = v/$\|\|$v$\|\|$
Two vectors **u** & **v** in RR^n, the distance between **u** & **v**	$\|\|$**u** - **v**$\|\|$
Two vectors **u** & **v** are orthogonal if and only if	$\|\|$u+v$\|\|^2$= $\|\|$u$\|\|^2$ +$\|\|$v$\|\|^2$ **u** • **v** = 0

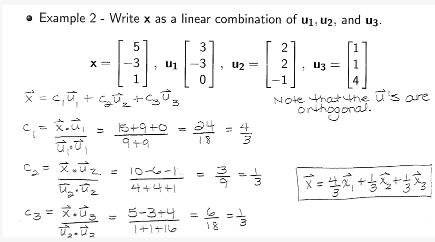
6.2

The distance from **y** to the line through **u** & the origin	$\|\|$z$\|\|$ = $\|\|$**y** - **y-hat**$\|\|$

6.2 Example (1)

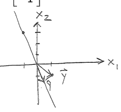


6.2 Example (2)



6.2 Example (3)

- Example 1 - Compute the orthogonal projection of $\begin{bmatrix}1\\-1\end{bmatrix}$ onto the line through $\begin{bmatrix}-1\\3\end{bmatrix}$ and the origin.



$\vec{y} = \begin{bmatrix}1\\-1\end{bmatrix}$ = vector being projected

$\vec{u} = \begin{bmatrix}-1\\3\end{bmatrix}$ = vector that the line passes through

orthogonal projection $= \hat{y} = \left(\frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)\vec{u}$

$= \left(\frac{-1-3}{(-1)^2+(3)^2}\right)\vec{u}$

$= \frac{-4}{10}\vec{u} = -\frac{2}{5}\vec{u} = \boxed{\begin{bmatrix}2/5\\-6/5\end{bmatrix}}$

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6.2 Example (4)

- Example 2 - Let $\mathbf{y} = \begin{bmatrix}2\\6\end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix}7\\1\end{bmatrix}$. Write $\mathbf{y}$ as the sum of a $\underbrace{\text{vector in Span}\{\mathbf{u}\}}_{=\alpha\vec{u}=\hat{y}}$ and a $\underbrace{\text{vector orthogonal to }\mathbf{u}}_{=\vec{z}}$.

$\vec{y} = \hat{y} + \vec{z}$

$\hat{y} = \left(\frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)\vec{u} = \left(\frac{14+6}{49+1}\right)\vec{u} = \frac{2}{5}\vec{u} = \begin{bmatrix}14/5\\2/5\end{bmatrix}$

$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix}2\\6\end{bmatrix} - \begin{bmatrix}14/5\\2/5\end{bmatrix} = \begin{bmatrix}-4/5\\28/5\end{bmatrix}$

$\boxed{\vec{y} = \begin{bmatrix}14/5\\2/5\end{bmatrix} + \begin{bmatrix}-4/5\\28/5\end{bmatrix}}$

6.2 Example (5)

- Example 3 - Let $\mathbf{y} = \begin{bmatrix}-3\\9\end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix}1\\2\end{bmatrix}$. Compute the distance from $\mathbf{y}$ to the line through $\mathbf{u}$ and the origin.

distance $= \|\vec{y} - \hat{y}\|$

① Find $\hat{y}$

$\hat{y} = \left(\frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)\vec{u} = \left(\frac{-3+18}{1+4}\right)\vec{u} = 3\vec{u} = \begin{bmatrix}3\\6\end{bmatrix}$

② Find $\vec{y} - \hat{y}$

$\vec{y} - \hat{y} = \begin{bmatrix}-3\\9\end{bmatrix} - \begin{bmatrix}3\\6\end{bmatrix} = \begin{bmatrix}-6\\3\end{bmatrix}$

③ Find $\|\vec{y} - \hat{y}\|$

$\|\vec{y} - \hat{y}\| = \sqrt{(-6)^2+(3)^2} = \sqrt{36+9} = \sqrt{45} = \boxed{3\sqrt{5}}$

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6.2 Example (6)

- Example 1 - Determine if the following set of vectors is orthonormal. If it is only orthogonal, normalize the vectors to produce an orthonormal set.

$\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\0\end{bmatrix}$ ($\vec{v}_1$, $\vec{v}_2$)

orthogonal and normalized

① Check if they are orthogonal

$\vec{v}_1\cdot\vec{v}_2 = 0-1+0 \neq 0$ => not orthogonal => not orthonormal

② No need to check if they are normalized since they are not orthogonal

6.2 Example (7)

- Example 2- Determine if the following set of vectors is orthonormal. If it is only orthogonal, normalize the vectors to produce an orthonormal set.

$\begin{bmatrix}-2/3\\1/3\\2/3\end{bmatrix}, \begin{bmatrix}1/3\\2/3\\0\end{bmatrix}$ ($\vec{v}_1$, $\vec{v}_2$)

orthogonal and normalized

① Check if they are orthogonal

$\vec{v}_1\cdot\vec{v}_2 = -2/9+2/9+0 = 0$ ✓

② Check if they are normalized

$\|\vec{v}_1\| = \sqrt{4/9+1/9+4/9} = 1$ ✓

$\|\vec{v}_2\| = \sqrt{1/9+4/9} = \frac{\sqrt{5}}{3} \neq 1$

$\frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{\vec{v}_2}{\sqrt{5}/3} = \frac{3}{\sqrt{5}}\vec{v}_2$

orthonormal set $= \left\{\vec{v}_1, \frac{\vec{v}_2}{\|\vec{v}_2\|}\right\}$

$= \boxed{\left\{\begin{bmatrix}-2/3\\1/3\\2/3\end{bmatrix}, \begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\\0\end{bmatrix}\right\}}$

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6.2 Reference (1)

- We end with the formula for the orthogonal projection of $\mathbf{y}$ onto L (or onto a line through $\vec{u}$ and the origin) and the component of $\mathbf{y}$ orthogonal to L:

memorize this!

$$\boxed{\hat{\mathbf{y}} = \left(\frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} \text{ and } \mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}}$$

- Note that $\hat{\mathbf{y}}$ is in Span$\{\mathbf{u}\}$ (because $\hat{y} = \alpha\vec{u}$, a linear combination of $\vec{u}$)

6.2 Reference (2)

$\underbrace{\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}}_{\text{orthogonal set}} \rightarrow \underbrace{\left\{\frac{\vec{v}_1}{\|\vec{v}_1\|}, \frac{\vec{v}_2}{\|\vec{v}_2\|}, \dots, \frac{\vec{v}_p}{\|\vec{v}_p\|}\right\}}_{\text{orthonormal set}}$

6.3 Example (1.1)

- Example - Assume that $\{\mathbf{u}_1, \cdots, \mathbf{u}_4\}$ is an orthogonal basis for $\mathbb{R}^4$. Write $\mathbf{x}$ as the sum of two vectors, $\underbrace{\text{one in Span }\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}}_{\vec{v}_1}$ and $\underbrace{\text{one in Span}\{\mathbf{u}_4\}}_{\vec{v}_2}$.

$$\mathbf{x} = \begin{bmatrix}4\\5\\-3\\3\end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix}-1\\1\\1\\-2\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}-2\\1\\-1\\1\end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix}1\\1\\-2\\-1\end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix}1\\2\\1\\1\end{bmatrix}$$

$\vec{x} = \vec{v}_1 + \vec{v}_2$

$\vec{v}_1$ = vector in Span $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\} = c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3$

$\vec{v}_2$ = vector in Span $\{\vec{u}_4\} = c_4\vec{u}_4$

Find c_1, c_2, c_3 and c_4 by using the fact that the $\vec{v}$'s are orthogonal.

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6.3 Example (1.2)

Find c_4 1st (because it's easier!)

$\vec{x} = \vec{v}_1 + \vec{v}_2 = c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 + c_4\vec{u}_4$

$\vec{x}\cdot\vec{u}_4 = c_1\underbrace{\vec{u}_1\cdot\vec{u}_4}_{0} + c_2\underbrace{\vec{u}_2\cdot\vec{u}_4}_{0} + c_3\underbrace{\vec{u}_3\cdot\vec{u}_4}_{0} + c_4\vec{u}_4\cdot\vec{u}_4$

$c_4 = \frac{\vec{x}\cdot\vec{u}_4}{\vec{u}_4\cdot\vec{u}_4} = \frac{4+10-3+3}{1+4+1+1} = \frac{14}{7} = 2$ => $\vec{v}_2 = c_4\vec{v}_4 = \begin{bmatrix}2\\4\\2\\2\end{bmatrix}$

To find $\vec{v}_1$ use the fact that

$\vec{x} = \vec{v}_1 + \vec{v}_2$

$\vec{v}_1 = \vec{x} - \vec{v}_2 = \begin{bmatrix}4\\5\\-3\\3\end{bmatrix} - \begin{bmatrix}2\\4\\2\\2\end{bmatrix} = \begin{bmatrix}2\\1\\-5\\1\end{bmatrix}$

$\vec{x} = \vec{v}_1 + \vec{v}_2 = \boxed{\begin{bmatrix}2\\1\\-5\\1\end{bmatrix} + \begin{bmatrix}2\\4\\2\\2\end{bmatrix}}$

6.3 Example (2)

- Example 1 - Verify that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set, and then find the orthogonal projection of $\mathbf{y}$ onto Span$\{\mathbf{u}_1, \mathbf{u}_2\}$.

$$\mathbf{y} = \begin{bmatrix}6\\3\\-2\end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix}3\\4\\0\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}-4\\3\\0\end{bmatrix}$$

$\vec{u}_1\cdot\vec{u}_2 = -12+12+0 = 0$ ✓ => $\vec{u}_1 \perp \vec{u}_2$

Orthogonal projection of $\vec{y}$ onto Span $\{\vec{u}_1, \vec{u}_2\}$

$= \hat{y} = \left(\frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\right)\vec{u}_1 + \left(\frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\right)\vec{u}_2$

$= \left(\frac{18+12+0}{9+16}\right)\vec{u}_1 + \left(\frac{-24+9+0}{16+9}\right)\vec{u}_2$

$= \frac{30}{25}\vec{u}_1 - \frac{15}{25}\vec{u}_2 = \frac{6}{5}\vec{u}_1 - \frac{3}{5}\vec{u}_2$

$= 6/5\begin{bmatrix}3\\4\\0\end{bmatrix} - 3/5\begin{bmatrix}-4\\3\\0\end{bmatrix}$

$= \boxed{\begin{bmatrix}6\\3\\0\end{bmatrix}}$

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6.3 Example (3)

- Example 3 - Let W be a subspace spanned by the $\mathbf{u}$'s, and write $\mathbf{y}$ as the sum of a $\underbrace{\text{vector in } W}_{\hat{y}}$ and a $\underbrace{\text{vector orthogonal to } W}_{\vec{z}}$.

$$\mathbf{y} = \begin{bmatrix}-1\\4\\3\end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}-1\\3\\-2\end{bmatrix}$$

$\vec{y} = \hat{y} + \vec{z}$

$\hat{y} = \left(\frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\right)\vec{u}_1 + \left(\frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\right)\vec{u}_2$

$\hat{y} = \left(\frac{-1+4+3}{1+1+1}\right)\vec{u}_1 + \left(\frac{1+12-6}{1+9+4}\right)\vec{u}_2$

$\hat{y} = 2\vec{u}_1 + \frac{1}{2}\vec{u}_2 = 2\begin{bmatrix}1\\1\\1\end{bmatrix} + 1/2\begin{bmatrix}-1\\3\\-2\end{bmatrix} = \begin{bmatrix}3/2\\7/2\\1\end{bmatrix}$

$\vec{z} = \vec{y} - \hat{y}$

$\vec{z} = \begin{bmatrix}-1\\4\\3\end{bmatrix} - \begin{bmatrix}3/2\\7/2\\1\end{bmatrix} = \begin{bmatrix}-5/2\\1/2\\2\end{bmatrix}$

$\boxed{\vec{y} = \begin{bmatrix}3/2\\7/2\\1\end{bmatrix} + \begin{bmatrix}-5/2\\1/2\\2\end{bmatrix}}$

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6.3 Example (4)

- Example 2 - Find the $\underbrace{\text{best approximation to } \mathbf{z}}_{=\hat{z}\text{ = closest point to }\vec{z}}$ by vectors of the form $\underbrace{c_1\mathbf{v}_1 + c_2\mathbf{v}_2}_{\text{Same as vectors in Span }\{\vec{v}_1, \vec{v}_2\}}$.

$$\mathbf{z} = \begin{bmatrix}2\\4\\0\\-1\end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix}2\\0\\-1\\-3\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}5\\-2\\4\\2\end{bmatrix}$$

$\hat{z} = \left(\frac{\vec{z}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1 + \left(\frac{\vec{z}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\right)\vec{v}_2$

$\hat{z} = \left(\frac{4+0+0+3}{4+0+1+9}\right)\vec{v}_1 + \left(\frac{10-8+0-2}{25+4+16+4}\right)\vec{v}_2$

$\hat{z} = \frac{1}{2}\vec{v}_1 + 0\vec{v}_2 = \boxed{\begin{bmatrix}1\\0\\-1/2\\-3/2\end{bmatrix}}$

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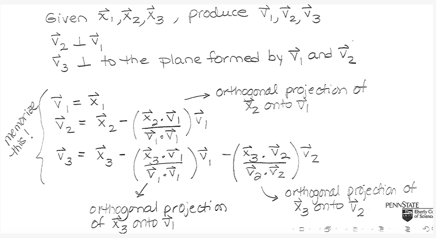
6.4

Gram- Schmidt Process Overview	take a given set of vectors & transform them into a set of orthogonal or orthonormal vectors
Given **x1** & **x2**, produce **v1** & **v2** where the v's are perp. to each other	(1) Let **v1=x1** (2) Find **v2**; **v2=x2 - x2hat**
x2 hat	(x2•v1)/(v1•v1) * v1

6.4 (cont)

Orthogonal Basis	{**v1**,**v2**,...,**vn**}
Orthonormal Basis	{**v1**/||**v1**||, **v2**/ ||**v2**||,..., **vn**/||**vn**||}

6.4 Reference (1)



Given $\vec{x}_1, \vec{x}_2, \vec{x}_3$, produce $\vec{v}_1, \vec{v}_2, \vec{v}_3$

$\vec{v}_2 \perp \vec{v}_1$

$\vec{v}_3 \perp$ to the plane formed by $\vec{v}_1$ and $\vec{v}_2$

memorize this!

$$\vec{v}_1 = \vec{x}_1$$
$$\vec{v}_2 = \vec{x}_2 - \left(\frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1$$
$$\vec{v}_3 = \vec{x}_3 - \left(\frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 - \left(\frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\right)\vec{v}_2$$

orthogonal projection of $\vec{x}_2$ onto $\vec{v}_1$

orthogonal projection of $\vec{x}_3$ onto $\vec{v}_1$

orthogonal projection of $\vec{x}_3$ onto $\vec{v}_2$

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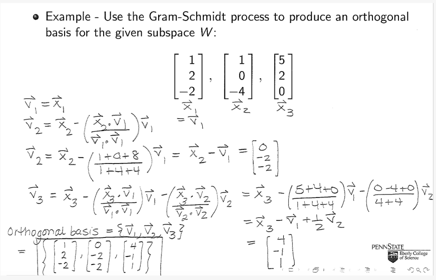
6.4 Example (1)



- Example - Use the Gram-Schmidt process to produce an orthogonal basis for the given subspace W:

$$\underset{\vec{x}_1 = \vec{v}_1}{\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}}, \underset{\vec{x}_2}{\begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}}, \underset{\vec{x}_3}{\begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}}$$

$\vec{v}_1 = \vec{x}_1$

$\vec{v}_2 = \vec{x}_2 - \left(\frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1$

$\vec{v}_2 = \vec{x}_2 - \left(\frac{1+0+8}{1+4+4}\right)\vec{v}_1 = \vec{x}_2 - \vec{v}_1 = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}$

$\vec{v}_3 = \vec{x}_3 - \left(\frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 - \left(\frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\right)\vec{v}_2 = \vec{x}_3 - \left(\frac{5+4+0}{1+4+4}\right)\vec{v}_1 - \left(\frac{0-4+0}{4+4}\right)\vec{v}_2$

$= \vec{x}_3 - \vec{v}_1 + \frac{1}{2}\vec{v}_2$

$= \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$

Orthogonal basis $= \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$= \boxed{\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} \right\}}$

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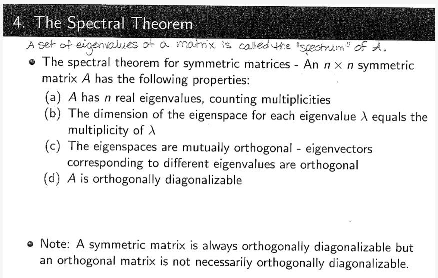
7.1

Symmetric Matrix	A square matrix where A^T=A
If A is a symmetric Matrix	then eigenvectors associated w/ distinct eigenvalues are orthogonal If a matrix is symmetrical, it has an orthogonal & orthonormal basis of vectors

7.1 (cont)

Orthogonal matrix is a square matrix w/ orthonormal columns	(1) Matrix is square (2) Columns are orthogonal (3) Columns are unit vectors
If Matrix P has orthonormal columns	P^TP=I
If P is a nxn orthogonal matrix	P^T=P^{-1}
A=PDP^T	A must be symmetric, P must be normalized

7.1 Reference (1)

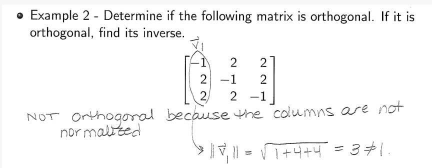


4. The Spectral Theorem

A set of eigenvalues of a matrix is called the "spectrum" of A.

- The spectral theorem for symmetric matrices - An $n \times n$ symmetric matrix A has the following properties:
  - (a) A has n real eigenvalues, counting multiplicities
  - (b) The dimension of the eigenspace for each eigenvalue λ equals the multiplicity of λ
  - (c) The eigenspaces are mutually orthogonal - eigenvectors corresponding to different eigenvalues are orthogonal
  - (d) A is orthogonally diagonalizable
- Note: A symmetric matrix is always orthogonally diagonalizable but an orthogonal matrix is not necessarily orthogonally diagonalizable.

7.1 Example (1)



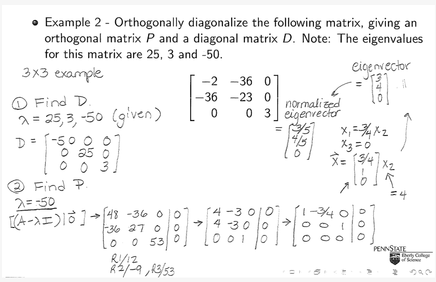
- Example 2 - Determine if the following matrix is orthogonal. If it is orthogonal, find its inverse.

$$\begin{array}{c} \vec{v}_1 \\ \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \end{array}$$

NOT orthogonal because the columns are not normalized

$\|\vec{v}_1\| = \sqrt{1+4+4} = 3 \neq 1$.

7.1 Example (2.1)



- Example 2 - Orthogonally diagonalize the following matrix, giving an orthogonal matrix P and a diagonal matrix D. Note: The eigenvalues for this matrix are 25, 3 and -50.

3x3 example

$$\begin{bmatrix} -2 & -36 & 0 \\ -36 & -23 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

① Find D.

$\lambda = 25, 3, -50$ (given)

$D = \begin{bmatrix} -50 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

② Find P.

$\lambda = -50$

$[(A-\lambda I)|\vec{0}] \to \left[\begin{array}{ccc|c} 48 & -36 & 0 & 0 \\ -36 & 27 & 0 & 0 \\ 0 & 0 & 53 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 4 & -3 & 0 & 0 \\ 4 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -3/4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

R1/12
R2/-9, R3/53

$x_1 = 3/4\, x_2$
$x_3 = 0$

$\vec{x} = \begin{bmatrix} 3/4 \\ 1 \\ 0 \end{bmatrix} x_2$, $= 4$

eigenvector $= \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$

normalized eigenvector $= \begin{bmatrix} 3/5 \\ 4/5 \\ 0 \end{bmatrix}$

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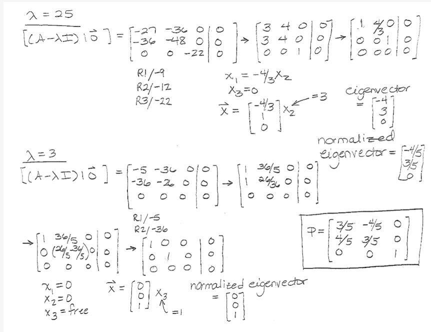
7.1 Example (2.2)



$\lambda = 25$

$[(A-\lambda I)|\vec{0}] = \left[\begin{array}{ccc|c} -27 & -36 & 0 & 0 \\ -36 & -48 & 0 & 0 \\ 0 & 0 & -22 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & 4 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

R1/-9
R2/-12
R3/-22

$x_1 = -4/3\, x_2$
$x_3 = 0$

$\vec{x} = \begin{bmatrix} -4/3 \\ 1 \\ 0 \end{bmatrix} x_2$, $= 3$

eigenvector $= \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$

normalized eigenvector $= \begin{bmatrix} -4/5 \\ 3/5 \\ 0 \end{bmatrix}$

$\lambda = 3$

$[(A-\lambda I)|\vec{0}] = \left[\begin{array}{ccc|c} -5 & -36 & 0 & 0 \\ -36 & -26 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 36/5 & 0 & 0 \\ 1 & 26/36 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

R1/-5
R2/-36

$\to \left[\begin{array}{ccc|c} 1 & 36/5 & 0 & 0 \\ 0 & (26/5 - 36/5) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 = 0$
$x_2 = 0$
$x_3 =$ free

$\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x_3$, $= 1$

normalized eigenvector $= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\boxed{P = \begin{bmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$