

## 1.1

A Matrix	row, columns
Coefficients Matrix	Just Left Hand Side
Augmented Matrix	Left and Right Hand Side
Solving Linear Systems	(1) Augmented Matrix (2) Row Operations (3) Solution to Linear System The RHS is the solution
One Solution	Upper triangle with Augmented Matrix
No Solution	Last row is all zeros = RHS number
Infinitely Many Solutions	Last row (including RHS) is all zeros
Inconsistent	Has No Solution

## 1.1 Example(1)

• Example - Use matrices to solve the following system of equations:

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

$$\left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array} \right] \xrightarrow{\substack{4R_1+R_3 \\ R_2/2 \text{ (scaling)}}} \left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array} \right] \xrightarrow{\substack{3R_2+R_3 \\ -R_2+R_1}} \left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right] \xrightarrow{\substack{2R_2+R_1 \\ -R_3+R_2 \\ -R_3+R_1}} \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 19 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

work down and to the right ↓  
then up and to the left ←

$$\begin{cases} x_1 = 29 \\ x_2 = 16 \\ x_3 = 3 \end{cases}$$

## 1.2

Echelon Matrix	(1) Zero Rows at the bottom (2) Leading Entries are down and to the right (3) Zeros are below each leading entry
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## 1.2 (cont)

Reduced Echelon Matrix	(1) The leading entry of each nonzero row is 1 (2) Zeros are below AND above each 1
Pivot Position	Location of Matrix that Corresponds to a leading 1 in REF
Pivot Column	Column in Matrix that contains a pivot
To get to EF	down and right
To get to REF	up and left
Free Variables	Variables that don't correspond to pivot columns
Consistent System	Pivot in every Column

## 1.2 Example (1)

• Example 1 - Determine the value(s) of h such that the following matrix is the augmented matrix of a consistent linear system.

$$\left[ \begin{array}{cc|c} 1 & -3 & 1 \\ h & 6 & -2 \end{array} \right]$$

Reduce the augmented matrix to echelon form

$$\left[ \begin{array}{cc|c} 1 & -3 & 1 \\ h & 6 & -2 \end{array} \right] \xrightarrow{hR_1 - R_2} \left[ \begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 6+3h & -h-2 \end{array} \right]$$

A consistent system cannot contain an equation of the form  $0 = \#$  but it can contain an equation of the form  $0 = 0$ .

Set  $6+3h=0$  and check the value of  $-h-2$

$$\begin{aligned} 3h &= -6 \\ h &= -2 \end{aligned}$$

The last row is  $0=0$  if  $h=-2$  and  $0=\#$  if  $h \neq -2$ . So is consistent for all h.

## 1.3

$RR^2$  Set of all vectors with 2 rows

## 1.3 Example (1)

• Example

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix}, \text{ and let } \mathbf{b} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$$

Determine if  $\mathbf{b}$  is a linear combination of  $\mathbf{a}_1, \mathbf{a}_2,$  and  $\mathbf{a}_3$ .

→ See if  $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$  has a solution by reducing  $[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ | \ \mathbf{b}]$  to echelon form and checking if it is consistent

$$\left[ \begin{array}{ccc|c} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 3 & 14 & 10 & -5 \end{array} \right] \xrightarrow{\substack{R_2/2 \\ -3R_1+R_3}} \left[ \begin{array}{ccc|c} 1 & 4 & 3 & -1 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 1 & -2 \end{array} \right] \xrightarrow{-2R_2+R_3} \left[ \begin{array}{ccc|c} 1 & 4 & 3 & -1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & -5 & -10 \end{array} \right]$$

pivot in each column  
system is consistent  
Yes,  $\mathbf{b}$  can be written as a linear combination of  $\mathbf{a}_1, \mathbf{a}_2,$  and  $\mathbf{a}_3$ .

## 1.3 Example (2)

• Example 1

$$\text{Let } \mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} h \\ -3 \\ -5 \end{bmatrix}$$

For what value(s) of h is  $\mathbf{b}$  in the plane spanned by  $\mathbf{a}_1$  and  $\mathbf{a}_2$ ?  
 $\mathbf{b}$  will be in the plane spanned by  $\mathbf{a}_1$  and  $\mathbf{a}_2$  if  $\mathbf{b}$  can be written as a linear combination of  $\mathbf{a}_1$  and  $\mathbf{a}_2$ .  
Reduce  $[\mathbf{a}_1 \ \mathbf{a}_2 \ | \ \mathbf{b}]$  to echelon form to determine what value(s) of h will make the system consistent.

$$\left[ \begin{array}{cc|c} 1 & -2 & h \\ 0 & 1 & -3 \\ -2 & 7 & -5 \end{array} \right] \xrightarrow{2R_1+R_3} \left[ \begin{array}{cc|c} 1 & -2 & h \\ 0 & 1 & -3 \\ 0 & 3 & 1 \end{array} \right] \xrightarrow{-3R_2+R_3} \left[ \begin{array}{cc|c} 1 & -2 & h \\ 0 & 1 & -3 \\ 0 & 0 & 1+h+9 \end{array} \right]$$

will be consistent if  $1+h+9=0$   
 $h = -10$

When  $h=-2$ ,  $\mathbf{b}$  is in the plane spanned by  $\mathbf{a}_1$  and  $\mathbf{a}_2$ .

## 1.4

Vector Equation  $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$

Matrix Equation  $A\mathbf{x} = \mathbf{b}$

If A is an m x n matrix the following are all true or all false

$A\mathbf{x} = \mathbf{b}$  has a solution for every  $\mathbf{b}$  in  $RR^m$

Every  $\mathbf{b}$  in  $RR^m$  is a lin. combo of columns in A

Columns of A span  $RR^m$

Matrix A has a pivot in every row (i.e. no row of zeros)

Anything in **Bold** means it is a vector.

## 1.4 Example (1)

• Example 2 - Show that the matrix equation  $A\mathbf{x} = \mathbf{b}$  does not have a solution for all possible  $\mathbf{b}$ , and describe the set of all  $\mathbf{b}$  for which  $A\mathbf{x} = \mathbf{b}$  does have a solution.

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 2 & 0 \\ 4 & -1 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Reduce  $[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ | \ \mathbf{b}]$  to echelon form to determine the restrictions on  $\mathbf{b}$  for the system to be consistent.

$$\left[ \begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ -2 & 2 & 0 & b_2 \\ 4 & -1 & 3 & b_3 \end{array} \right] \xrightarrow{\substack{2R_1+R_2 \\ -4R_1+R_3}} \left[ \begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & -2 & -2 & 2b_1+b_2 \\ 0 & 7 & 7 & 7b_1+b_3 \end{array} \right] \xrightarrow{R_2/(-2)} \left[ \begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & 1 & 1 & -b_1-b_2/2 \\ 0 & 7 & 7 & 7b_1+b_3 \end{array} \right]$$

A restriction on  $\mathbf{b}$  will occur when the last row does not have a pivot.

This system will only have a solution when  $b_3 - 4b_1 + \frac{7}{2}b_2 + 7b_1 = 0 \Rightarrow 3b_1 + \frac{7}{2}b_2 + b_3 = 0$



## 1.4 Example (2)

• Example - Determine if the columns of matrix  $A$  span  $\mathbb{R}^3$ .  
ie. Determine if  $A\vec{x}=\vec{b}$  has a solution for all  $\vec{b}$ .  
Reduce  $[\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3 \ | \ \vec{b}]$  to echelon form.

$$A = \begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -2 \\ -3 & 9 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 4 & | & b_1 \\ 0 & -3 & -2 & | & b_2 \\ -3 & 9 & -6 & | & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} -3 & 9 & -6 & | & b_3 \\ 0 & -3 & -2 & | & b_2 \\ 0 & 0 & 4 & | & b_1 \end{bmatrix}$$

Every row has a pivot so there will be no restrictions on  $\vec{b}$  and there will be a solution for every  $\vec{b}$ .  
This means that the columns of  $A$  span  $\mathbb{R}^3$  because there are 3 rows in  $A$ .

## 1.5

Homogeneous  $A\vec{x} = \vec{0}$

Trivial Solution  $A\vec{x} = \vec{0}$  if at least one column is missing a pivot

Determine if homogenous Linear System has a non trivial solution

- (1) Write as Augmented Matrix
- (2) Reduce to EF
- (3) Determine if there are any free variables- (column w/o pivot)
- (4) If any free variables, than a non-trivial solution exists
- (5) Non-Trivial Solution can be found by further reducing to REF and solving for  $\vec{x}$

If  $A\vec{x} = \vec{0}$  has one free variable Than  $\vec{x}$  is a line that passes through the origin

If  $A\vec{x} = \vec{0}$  has two free variables Than  $\vec{x}$  has a plane that passes through the origin

## 1.5 Example (1)

• Example 1 - Determine if the following linear system has a nontrivial solution and then describe the solution set.

$$\begin{cases} x_1 - 2x_2 + 3x_3 = 0 \\ -2x_1 - 3x_2 - 4x_3 = 0 \\ 2x_1 - 4x_2 + 9x_3 = 0 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & 3 & | & 0 \\ -2 & -3 & -4 & | & 0 \\ 2 & -4 & 9 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 & | & 0 \\ 0 & -1 & -10 & | & 0 \\ 0 & 0 & 15 & | & 0 \end{bmatrix}$$

Every column has a pivot so there are no free variables.  
 $\vec{x} = \vec{0}$  (the trivial solution) is the only solution.  
No nontrivial solution.

## 1.5 Example (2)

• Example 2 - Determine if the following linear system has a nontrivial solution and then describe the solution set.

$$\begin{cases} 2x_1 + 4x_2 - 6x_3 = 0 \\ 4x_1 + 8x_2 - 10x_3 = 0 \end{cases}$$

$$\begin{bmatrix} 2 & 4 & -6 & | & 0 \\ 4 & 8 & -10 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & -6 & | & 0 \\ 0 & 0 & 2 & | & 0 \end{bmatrix}$$

Column 2 is missing a pivot  $\Rightarrow x_2$  is free variable and  $\vec{x}$  has a nontrivial solution.

Reduce  $[A \ | \ \vec{0}]$  to reduced echelon form to solve for  $\vec{x}$ .

$$\begin{bmatrix} 2 & 4 & -6 & | & 0 \\ 0 & 0 & 2 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$x_1 = -2x_2$   
 $x_2 = \text{free}$   
 $x_3 = 0$

## 1.7

Linear Independence To check ind/dep

No free Variables, none of the vectors are multiples of each other

reduce augmented matrix to EF and see if there are free variables (ie. every column must have a pivot to be linearly independent)

To check if multiples

$\vec{u} = c * \vec{v}$   
find value of  $c$ , then it is a multiple therefore linearly dependent

Linearly Dependent If there are more columns than rows

## 1.7 Example (1)

• Example 4 - Determine the values of  $h$  that make the following vectors linearly dependent.

$$\vec{v}_1 = \begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 9 \\ h \\ 3 \end{bmatrix}$$

Reduce  $[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ | \ 0]$  to echelon form and choose  $h$  so that there is a free variable.

$$\begin{bmatrix} 3 & -6 & 9 & | & 0 \\ -6 & 4 & h & | & 0 \\ 1 & -3 & 3 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 & | & 0 \\ -6 & 4 & h & | & 0 \\ 3 & -6 & 9 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 & | & 0 \\ 0 & -7 & h+18 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$-R_2/R_2 + R_3$

$$\rightarrow \begin{bmatrix} 1 & -2 & 3 & | & 0 \\ 0 & -7 & h+18 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

For there to be a free variable,  $-(h+18) = 0 \Rightarrow h = -18$

## 1.8

Every Matrix Transformation is a: Linear Transformation

$T(\vec{x}) = A(\vec{x})$

If  $A$  is  $m \times n$  Matrix, then the properties are

- (1)  $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
- (2)  $T(c\vec{u}) = cT(\vec{u})$
- (3)  $T(\vec{0}) = \vec{0}$
- (4)  $T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$

## 1.8 Example (1)

• Example 5 - Suppose  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a linear transformation such that  $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$  and  $T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ .  
Find  $T\left(\begin{bmatrix} 4 \\ 0 \end{bmatrix}\right)$ . Let  $\vec{x} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ .

① write  $\vec{x}$  as a linear combination of  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ .

$$\begin{bmatrix} 4 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{cases} 4 = c_1 + c_2 \\ 0 = c_1 - c_2 \end{cases} \Rightarrow c_1 = 2 \ \& \ c_2 = 2$$

② Find the transformation of  $\vec{x}$

$$T\left(\begin{bmatrix} 4 \\ 0 \end{bmatrix}\right) = T\left(2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = 2T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + 2T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = 2\begin{bmatrix} 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

given in the original problem.

## 1.8 Example (2)

• Example 5 - Suppose  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a linear transformation such that  $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$  and  $T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ .  
Find  $T\left(\begin{bmatrix} 4 \\ 0 \end{bmatrix}\right)$ . Let  $\vec{x} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ .

① write  $\vec{x}$  as a linear combination of  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ .

$$\begin{bmatrix} 4 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{cases} 4 = c_1 + c_2 \\ 0 = c_1 - c_2 \end{cases} \Rightarrow c_1 = 2 \ \& \ c_2 = 2$$

② Find the transformation of  $\vec{x}$

$$T\left(\begin{bmatrix} 4 \\ 0 \end{bmatrix}\right) = T\left(2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = 2T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + 2T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = 2\begin{bmatrix} 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

given in the original problem.

### 1.9

$\mathbb{R}R^n \rightarrow \mathbb{R}R^m$  Equation  $T(\mathbf{x}) = A\mathbf{x} = \mathbf{b}$  has a unique solution or more than one solution each row has a pivot

$\mathbb{R}R^n \rightarrow \mathbb{R}R^m$  Equation  $T(\mathbf{x}) = A\mathbf{x} = \mathbf{b}$  has a unique solution or no solution each row has a pivot

### 2.1

**Addition of Matrices** Can Add matrices if they have same # of rows and columns (ie  $A(3 \times 4)$  and  $B(3 \times 4)$  so you can add them)

**Multiply by Scalar** Multiply each entry by scalar

**Matrix Multiplication ( $A \times B$ )** Must each row of  $A$  by each column of  $B$

**Powers of a Matrix** Can compute powers by if the matrix has the same number of columns as rows

**Transpose of Matrix** row 1 of  $A$  becomes column 1 of  $A$   
row 2 of  $A$  becomes column 2 of  $A$

### 2.1 (cont)

**Properties of Transpose**

- (1) if  $A$  is  $m \times n$ , then  $A^T$  is  $n \times m$
- (2)  $(A^T)^T = A$
- (3)  $(A + B)^T = A^T + B^T$
- (4)  $(tA)^T = tA^T$
- (5)  $(AB)^T = B^T A^T$

### 2.2

**Singular matrix** A matrix that is NOT invertible

**Determinate of  $A(2 \times 2)$  Matrix**  $\det A = ad - bc$

**If  $A$  is invertible &  $(n \times n)$**  There will never be no solution or infinitely many solutions to  $A\mathbf{x} = \mathbf{b}$

**Properties of Invertible Matrices**

- $(A^{-1})^{-1} = A$
- (assuming  $A$  &  $B$  are invertible)  $(AB)^{-1} = B^{-1} A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$

**Finding Inverse Matrix**  $[A | I] \rightarrow [I | A^{-1}]$  Use row operations  
STOP when you get a row of Zeros, it cannot be reduced

### 2.2 Example (1)

• Example 2 - Let  $A, B, C$  and  $X$  be  $n \times n$  invertible matrices. Solve  $B(X + A)^{-1} = C$  for the matrix  $X$ .

**Method 1**  
Set  $(X+A)^{-1}$  by itself 1st.  
 $B(X+A)^{-1} = C$   
 $B^{-1}B(X+A)^{-1} = B^{-1}C$   
Note: If you multiply on the left for one side, you must multiply on the left on the other side.  
 $I(X+A)^{-1} = B^{-1}C$   
 $(X+A)^{-1} = B^{-1}C$   
 $((X+A)^{-1})^{-1} = (B^{-1}C)^{-1}$   
 $X+A = C^{-1}(B^{-1})^{-1}$   
 $X = C^{-1}B - A$

**Method 2**  
 $B(X+A)^{-1} = C$   
 $(B(X+A)^{-1})^{-1} = C^{-1}$   
 $((X+A)^{-1})^{-1} B^{-1} = C^{-1}$   
 $(X+A)B^{-1} = C^{-1}$   
 $X B^{-1} + A B^{-1} = C^{-1}$   
 $X B^{-1} = C^{-1} - A B^{-1}$   
 $X B^{-1} B = (C^{-1} - A B^{-1}) B$   
 $X = C^{-1} B - A B^{-1} B$   
 $X = C^{-1} B - A$

### 2.3 Invertible Matrix Theorem

- The Invertible Matrix Theorem - Let  $A$  be a square  $n \times n$  matrix. Then all of the following statements are equivalent:
  - (a)  $A$  is an invertible matrix
  - (b)  $A$  is row equivalent to the  $n \times n$  identity matrix  $I$ .
  - (c)  $A$  has  $n$  pivots.
  - (d) The equation  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution.
  - (e) The columns of  $A$  form a linearly independent set.
  - (f) The linear transformation  $T(\mathbf{x}) = A\mathbf{x}$  is one-to-one.
  - (g) The equation  $A\mathbf{x} = \mathbf{b}$  has a unique solution for each  $\mathbf{b}$  in  $\mathbb{R}^n$ .
  - (h) The columns of  $A$  span  $\mathbb{R}^n$ .
  - (i) The linear transformation  $T(\mathbf{x}) = A\mathbf{x}$  is onto.
  - (j) There is an  $n \times n$  matrix  $C$  such that  $CA = I$ .
  - (k) There is an  $n \times n$  matrix  $D$  such that  $AD = I$ .
  - (l)  $A^T$  is invertible.

The above theorem states that if one of these is false, they all must be false. If one is true, then they are all true.

### 2.8

**A subspace  $S$  of  $\mathbb{R}R^n$  is a subspace is  $S$  satisfies:**

- (1)  $S$  contains zero vector
- (2) If  $\mathbf{u}$  &  $\mathbf{v}$  are in  $S$ , then  $\mathbf{u} + \mathbf{v}$  is also in  $S$
- (3) If  $r$  is a real # &  $\mathbf{u}$  is in  $S$ , then  $r\mathbf{u}$  is also in  $S$

**Subspace  $\mathbb{R}R^3$**  Any Plane that Passes through the origin forms a subspace  $\mathbb{R}R^3$   
Any set that contains nonlinear terms will NOT form a subspace  $\mathbb{R}R^3$

**Null Space (Nul  $A$ )** To determine in  $\mathbf{u}$  is in the Nul( $A$ ), check if:  $A\mathbf{u} = \mathbf{0}$   
If yes  $\rightarrow$  then  $\mathbf{u}$  is in the Nullspace



## 2.8 Example (1)

- Example 1 - Given the following matrix A and an echelon form of A, find a basis for Col A.

$$A = \begin{bmatrix} 3 & -6 & 9 & 0 \\ 2 & -4 & 7 & 2 \\ 3 & -6 & 6 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 & 4 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

echelon form of A.  
pivot columns = columns 1 and 3  
basis for Col A = pivot columns of A (not pivot columns of echelon form of A)  
basis for Col A =  $\left\{ \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 9 \\ 7 \\ 6 \end{bmatrix} \right\}$   
Col A =  $c_1 \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 9 \\ 7 \\ 6 \end{bmatrix}$

## 2.8 Example (2)

- Example 2 - Given the following matrix A and an echelon form of A, find a basis for Nul A.

$$A = \begin{bmatrix} 3 & -6 & 9 & 0 \\ 2 & -4 & 7 & 2 \\ 3 & -6 & 6 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 & 4 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduce [A|0] to reduced echelon form to solve for  $\vec{x}$   
 $\begin{bmatrix} 1 & -2 & 5 & 4 & 0 \\ 0 & 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 5 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$   
 $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_2 + 6x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 6 \\ 0 \\ -1 \\ 1 \end{bmatrix}$   
Basis for Nul A =  $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$  \*vectors in basis for Nul A = # free variables.

## 2.9

Dimension of a non-zero Subspace # of vectors in any basis; it is the # of linearly independent vectors

Dimension of a zero Subspace is Zero

Dimension of a Column Space # of pivot columns

Dimension of a Null Space # of free variables in the solution  $Ax=0$

Rank of a Matrix # of pivot columns

The Rank Theorem Matrix A has n columns: rank A (# pivots) + dim Nul A (# free var.) = n

dim = dimension; var. = variable

## 2.9 Reference

- The Invertible Matrix Theorem Continued- Let A be a square  $n \times n$  matrix. Then all of the following statements are equivalent to the statement that A is an invertible matrix. (See Section 2.3)
    - (m) The columns of A form a basis for  $\mathbb{R}^n$  (because they are linearly independent)
    - (n) Col A =  $\mathbb{R}^n$
    - (o) dim Col A = n (because n pivots)
    - (p) rank A = n (because n pivots)
    - (q) Nul A = {0} (because no free variables)
    - (r) dim Nul A = 0 (because no free variables)
- Because every vector in  $\mathbb{R}^n$  can be written as a linear combination of the columns of A. rank A + dim Nul A = n

## 3.1

Calculating Determinant of Matrix A is another way to tell if a linear system of equations has a solution

(1) Det(A) not = 0, then  $Ax=b$  has a unique solution

(2) Det(A) = 0, then  $Ax=b$  has no solutions or inf many

If  $Ax \neq 0$

$A^{-1}$  exist

If  $Ax = 0$

$A^{-1}$  Does NOT exist

Cofactor Expansion

Use row/column w/ most zeros

If Matrix A has an upper or lower triangle of zeros

The det(A) is the multiplication down the diagonals

## 3.1 Reference (1)

### 2. Determinate of a $3 \times 3$ Matrix

- Let A be an  $n \times n$  matrix (or here a  $3 \times 3$  matrix)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- The determinate of matrix A is given as follows:

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$$

$$\det A_{11} = a_{22}a_{33} - a_{23}a_{32} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\det A_{12} = a_{21}a_{33} - a_{23}a_{31} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$\det A_{13} = a_{21}a_{32} - a_{22}a_{31} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

\* This formula utilizes a cofactor expansion across the first row

## 3.1 Example (1)

- Example - Compute the determinate of

$$A = \begin{bmatrix} -3 & 1 & 2 \\ 5 & 5 & -8 \\ 4 & 2 & -5 \end{bmatrix}$$

$$\det A = \begin{vmatrix} -3 & 1 & 2 \\ 5 & 5 & -8 \\ 4 & 2 & -5 \end{vmatrix} = +(-3) \begin{vmatrix} 5 & -8 \\ 2 & -5 \end{vmatrix} - 1 \begin{vmatrix} 5 & -8 \\ 4 & -5 \end{vmatrix} + 2 \begin{vmatrix} 5 & 5 \\ 4 & 2 \end{vmatrix}$$

$$= -3(-25+16) - 1(-25+32) + 2(10-20)$$

$$= -3(-9) - 1(7) + 2(-10)$$

$$= 27 - 7 - 20$$

$$= 0$$

← This means  $A^{-1}$  does not exist and  $Ax=b$  has no solution or infinitely many solutions.

## 3.1 Reference (2)

- Shortcut Method for a  $3 \times 3$  matrix

add column 1 + 2 to the right of the matrix.

$$\det = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31}$$

## 3.1 Example (2)

- Example 2 - Use a cofactor expansion to compute the determinate of

$$A = \begin{bmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{bmatrix}$$

Choose the row or column with the most zeros  $\Rightarrow$  choose row 2.

$$\det A = -3 \begin{vmatrix} 1 & -2 & 2 \\ 2 & -6 & 5 \\ 5 & 0 & 4 \end{vmatrix} = -3 \left[ +5 \begin{vmatrix} -2 & 2 \\ -6 & 5 \end{vmatrix} - 0 \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} + 10 \begin{vmatrix} 1 & -2 \\ 2 & -6 \end{vmatrix} \right]$$

$$= -3 \left[ 5(-10+12) - 0 + 4(-6+4) \right]$$

$$= -3(10-8) = -6$$

You can also use the shortcut 3x3 method to evaluate this.

## 3.2

Determinate Property 1 If a multiple of 1 row of A is added to another row to produce Matrix B, then  $\det(B) = \det(A)$

Determinate Property 2 If 2 rows of A are interchanged to produce B, then  $\det(B) = -\det(A)$

Determinate Property 3 If one row of A is multiplied to produce B, then  $\det(B) = k \cdot \det(A)$



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Published 22nd April, 2018.  
Last updated 22nd April, 2018.  
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### 3.2 (cont)

Assuming both A & B are  $n \times n$  Matrices

- (1)  $\det(A^T) = \det(A)$
- (2)  $\det(AB) = \det(A) \cdot \det(B)$
- (3)  $\det(A^{-1}) = 1/\det(A)$
- (4)  $\det(cA) = c^n \det(A)$
- (5)  $\det(A^T) = \det(A)^r$

### 3.3 AKA Cramer's Rule

**Cramer's Rule** Can be used to find the solution to a linear system of equations  $Ax=b$  when A is an invertible square matrix

**Def. of Cramer's Rule** Let A be an  $n \times n$  invertible matrix. For any  $b$  in  $\mathbb{R}^n$ , the unique solution  $x$  of  $Ax=b$  has entries given by  $x_i = \det(A_i(b))/\det(A)$   $i = 1, 2, \dots, n$

$A_i(b)$  is the matrix A w/ column  $i$  replaced w/ vector  $b$

### 3.3 Example (1)

• Example 2 - Use Cramer's Rule to compute the solution to the system

Use to use the shortcut method for 3x3's.

$$\begin{cases} 3x_1 + x_2 = 5 \\ -x_1 + 2x_2 + x_3 = -2 \\ -x_2 + 2x_3 = -1 \end{cases}$$

$x_1 = \frac{\begin{vmatrix} 5 & 1 & 0 \\ -2 & 2 & 1 \\ -1 & -1 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & 0 \\ -1 & 2 & 1 \\ 0 & -1 & 2 \end{vmatrix}} = \frac{20 - 1 + 0 + 4 + 5 - 0}{12 + 0 + 0 + 2 + 3 - 0} = \frac{28}{17}$

$x_2 = \frac{\begin{vmatrix} 3 & 5 & 0 \\ -1 & -2 & 1 \\ 0 & -1 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & 0 \\ -1 & 2 & 1 \\ 0 & -1 & 2 \end{vmatrix}} = \frac{-9 + 0 + 0 + 0 + 3 - 0}{17} = \frac{-8}{17}$

$x_3 = \frac{\begin{vmatrix} 3 & 1 & 5 \\ -1 & 2 & -2 \\ 0 & -1 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & 0 \\ -1 & 2 & 1 \\ 0 & -1 & 2 \end{vmatrix}} = \frac{-9 + 0 + 5 + 1 - 0 - 0}{17} = \frac{-3}{17}$

### 5.1

$Au = \lambda u$  A is an  $n \times n$  matrix. A nonzero vector  $u$  is an eigenvector of A if there exists such a scalar  $\lambda$

To determine if  $\lambda$  is an eigenvalue

reduce  $[(A-\lambda I)]0$  to echelon form and see if it has any free variables.

yes ->  $\lambda$  is Eigenvalue  
no ->  $\lambda$  is not eigenvalue

To determine if given vector is an eigenvector

$Ax = \lambda x$

Eigenspace of A = Nullspace of  $(A-\lambda I)$

Eigenvalues of triangular Matrix entries along diagonal \*you CANNOT row reduce a matrix to find its eigenvalues

### 5.1 Example (1)

• Example 2 (3x3 Example)

Is  $\lambda = 1$  an eigenvalue of  $A = \begin{bmatrix} 4 & -2 & 3 \\ 0 & -1 & 3 \\ -1 & 2 & -2 \end{bmatrix}$ ? Why or why not?

Also, find the corresponding eigenvector.

Check if  $[(A-\lambda I)]0$  has a nontrivial solution (free variable)

$$[(A-\lambda I)]0 = \begin{bmatrix} 3 & -2 & 3 \\ 0 & -2 & 3 \\ -1 & 2 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & -2 & 3 & | & 0 \\ 0 & -2 & 3 & | & 0 \\ -1 & 2 & -5 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & -2 & 3 & | & 0 \\ 0 & -2 & 3 & | & 0 \\ 0 & 0 & -2 & | & 0 \end{bmatrix}$$

$x_3 = \text{free var.}$

$x_2 = \frac{3}{2}x_3$

$x_1 = 0$

$\vec{x} = \begin{bmatrix} 0 \\ \frac{3}{2}x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ \frac{3}{2} \\ 1 \end{bmatrix}$

Choose  $x_3 = 2$

one eigenvector  $\vec{v} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$

$\Rightarrow \lambda = 1$  is an eigenvalue

### 5.1 Example (2)

• Example 2

Is  $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$  an eigenvector of  $A = \begin{bmatrix} 3 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix}$ ? If so, find the eigenvalue.

Check if  $A\vec{x} = \lambda\vec{x}$

$$A\vec{x} = \begin{bmatrix} 3 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 1 \end{bmatrix} \stackrel{?}{=} \lambda \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

No value of  $\lambda$  will make this true  
 $\vec{x}$  is not an eigenvector.

### 5.1 Example (3)

matrix A and eigenvalue  $\lambda$ .

Reduce  $[(A-\lambda I)]0$  to reduced echelon form and solve for  $\vec{x}$ .

$A = \begin{bmatrix} 4 & 4 & -2 \\ 1 & 4 & -1 \\ 3 & 6 & -1 \end{bmatrix}, \lambda = 2$

$$[(A-\lambda I)]0 = \begin{bmatrix} 2 & 2 & -4 \\ 1 & 4 & -1 \\ 3 & 6 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2 & -4 & | & 0 \\ 1 & 4 & -1 & | & 0 \\ 3 & 6 & -3 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -2 & | & 0 \\ 1 & 4 & -1 & | & 0 \\ 3 & 6 & -3 & | & 0 \end{bmatrix}$$

$x_1 = -2x_2 + x_3$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

basis for eigenspace =  $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$  ← geometric description is a plane

### 5.2

If  $\lambda$  is an eigenvalue of a Matrix A then  $(A-\lambda I)x=0$  will have a nontrivial solution

A nontrivial solution will exist if  $\det(A-\lambda I) = 0$  (Characteristic Equation)

A is  $n \times n$  Matrix. A is invertible if and only if

- (1) The  $\# 0$  is NOT an  $\lambda$  of A
- (2) The  $\det(A)$  is not zero

Similar Matrices If  $n \times n$  Matrices A and B are similar, then they have the same characteristic polynomial (same  $\lambda$ ) with same multiplicities





## 6.2 Example (3)

- Example 1 - Compute the orthogonal projection of  $\vec{y} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$  onto the line through  $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$  and the origin.

$\vec{y} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$  = vector being projected  
 $\vec{u} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$  = vector that the line passes through

orthogonal projection =  $\hat{y} = \frac{(\vec{y} \cdot \vec{u})}{\vec{u} \cdot \vec{u}} \vec{u}$   
 $= \frac{(-1)(-1) + (-3)(-1)}{(-1)^2 + (3)^2} \vec{u}$   
 $= \frac{-4}{10} \vec{u} = -\frac{2}{5} \vec{u} = \begin{bmatrix} 2/5 \\ -6/5 \end{bmatrix}$

## 6.2 Example (4)

- Example 2 - Let  $\vec{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$  and  $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . Write  $\vec{y}$  as the sum of a vector in  $\text{Span}(\vec{u})$  and a vector orthogonal to  $\vec{u}$ .

$\vec{y} = \vec{y} + \vec{z}$   
 $\hat{y} = \frac{(\vec{y} \cdot \vec{u})}{\vec{u} \cdot \vec{u}} \vec{u} = \frac{(2+6)}{1+1} \vec{u} = \frac{8}{2} \vec{u} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$   
 $\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$   
 $\vec{y} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

## 6.2 Example (5)

- Example 3 - Let  $\vec{y} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$  and  $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ . Compute the distance from  $\vec{y}$  to the line through  $\vec{u}$  and the origin.

distance =  $\|\vec{y} - \hat{y}\|$

- Find  $\hat{y}$   
 $\hat{y} = \frac{(\vec{y} \cdot \vec{u})}{\vec{u} \cdot \vec{u}} \vec{u} = \frac{(-3+18)}{1+4} \vec{u} = 3 \vec{u} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$
- Find  $\vec{y} - \hat{y}$   
 $\vec{y} - \hat{y} = \begin{bmatrix} -3 \\ 9 \end{bmatrix} - \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$
- Find  $\|\vec{y} - \hat{y}\|$   
 $\|\vec{y} - \hat{y}\| = \sqrt{(-6)^2 + (3)^2} = \sqrt{36+9} = \sqrt{45} = 3\sqrt{5}$

## 6.2 Example (6)

- Example 1 - Determine if the following set of vectors is orthonormal. If it is only orthogonal, normalize the vectors to produce an orthonormal set.

$\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$  orthogonal and normalized

- Check if they are orthogonal  
 $\vec{v}_1 \cdot \vec{v}_2 = 0(-1) + 0(0) = 0 \Rightarrow$  not orthogonal  $\Rightarrow$  not orthonormal
- No need to check if they are normalized since they are not orthogonal

## 6.2 Example (7)

- Example 2 - Determine if the following set of vectors is orthonormal. If it is only orthogonal, normalize the vectors to produce an orthonormal set.

$\vec{v}_1 = \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix}$  orthogonal and normalized

- Check if they are orthogonal  
 $\vec{v}_1 \cdot \vec{v}_2 = -2/9 + 2/9 + 0 = 0 \checkmark$
- Check if they are normalized  
 $\|\vec{v}_1\| = \sqrt{4/9 + 1/9 + 4/9} = \sqrt{9/9} = 1 \checkmark$   
 $\|\vec{v}_2\| = \sqrt{1/9 + 4/9 + 0} = \sqrt{5/9} \neq 1$   
 $\frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{\vec{v}_2}{\sqrt{5}/3} = \frac{3}{\sqrt{5}} \vec{v}_2$

orthonormal set =  $\left\{ \vec{v}_1, \frac{3}{\sqrt{5}} \vec{v}_2 \right\}$

## 6.2 Reference (1)

- We end with the formula for the orthogonal projection of  $\vec{y}$  onto  $L$  and the component of  $\vec{y}$  orthogonal to  $L$ :

memorize this!  
 $\hat{y} = \frac{(\vec{y} \cdot \vec{u})}{\vec{u} \cdot \vec{u}} \vec{u}$  and  $\vec{z} = \vec{y} - \hat{y}$  (or onto a line through  $\vec{u}$  and the origin)

- Note that  $\vec{y}$  is in  $\text{Span}(\vec{u})$  because  $\vec{y} = \alpha \vec{u}$  (a linear combination of  $\vec{u}$ )

## 6.2 Reference (2)

$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\} \rightarrow \left\{ \frac{\vec{v}_1}{\|\vec{v}_1\|}, \frac{\vec{v}_2}{\|\vec{v}_2\|}, \dots, \frac{\vec{v}_p}{\|\vec{v}_p\|} \right\}$   
 orthogonal set  $\rightarrow$  orthonormal set

## 6.3 Example (1.1)

- Example - Assume that  $\{\vec{u}_1, \dots, \vec{u}_4\}$  is an orthogonal basis for  $\mathbb{R}^4$ . Write  $\vec{x}$  as the sum of two vectors, one in  $\text{Span}(\vec{u}_1, \vec{u}_2, \vec{u}_3)$  and one in  $\text{Span}(\vec{u}_4)$ .

$\vec{x} = \begin{bmatrix} 5 \\ 4 \\ -3 \\ 3 \end{bmatrix}, \vec{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 1 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$

$\vec{x} = \vec{v}_1 + \vec{v}_2$   
 $\vec{v}_1 =$  vector in  $\text{Span}(\vec{u}_1, \vec{u}_2, \vec{u}_3)$   $\vec{v}_2 = c_4 \vec{u}_4$   
 $\vec{v}_2 =$  vector in  $\text{Span}(\vec{u}_4)$   $\vec{v}_2 = c_4 \vec{u}_4$   
 Find  $c_1, c_2, c_3$  and  $c_4$  by using the fact that the  $\vec{v}$ 's are orthogonal.

## 6.3 Example (1.2)

Find  $c_4$  list (because it's easier!)  
 $\vec{x} = \vec{v}_1 + \vec{v}_2 = c_1 \vec{u}_1 + c_2 \vec{u}_2 + c_3 \vec{u}_3 + c_4 \vec{u}_4$   
 $\vec{x} \cdot \vec{u}_4 = c_1 \vec{u}_1 \cdot \vec{u}_4 + c_2 \vec{u}_2 \cdot \vec{u}_4 + c_3 \vec{u}_3 \cdot \vec{u}_4 + c_4 \vec{u}_4 \cdot \vec{u}_4$   
 $c_4 = \frac{\vec{x} \cdot \vec{u}_4}{\vec{u}_4 \cdot \vec{u}_4} = \frac{4+10-3+3}{1+4+1+1} = \frac{14}{7} = 2 \Rightarrow \vec{v}_2 = c_4 \vec{u}_4 = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2 \end{bmatrix}$   
 To find  $\vec{v}_1$ , use the fact that  $\vec{x} = \vec{v}_1 + \vec{v}_2$   
 $\vec{v}_1 = \vec{x} - \vec{v}_2 = \begin{bmatrix} 5 \\ 4 \\ -3 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -5 \\ 1 \end{bmatrix}$   
 $\vec{x} = \vec{v}_1 + \vec{v}_2 = \begin{bmatrix} 3 \\ 0 \\ -5 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2 \end{bmatrix}$

## 6.3 Example (2)

- Example 1 - Verify that  $\{\vec{u}_1, \vec{u}_2\}$  is an orthogonal set, and then find the orthogonal projection of  $\vec{y}$  onto  $\text{Span}(\vec{u}_1, \vec{u}_2)$ .

$\vec{y} = \begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix}, \vec{u}_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$   
 $\vec{u}_1 \cdot \vec{u}_2 = -12 + 12 + 0 = 0 \checkmark \Rightarrow \vec{u}_1 \perp \vec{u}_2$   
 Orthogonal projection of  $\vec{y}$  onto  $\text{Span}(\vec{u}_1, \vec{u}_2)$   
 $= \hat{y} = \frac{(\vec{y} \cdot \vec{u}_1)}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \frac{(\vec{y} \cdot \vec{u}_2)}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2 = \frac{6(3) + 3(4)}{9+16} \vec{u}_1 + \frac{(-4)(3) + 3(0)}{16+9} \vec{u}_2$   
 $= \frac{30}{25} \vec{u}_1 - \frac{12}{25} \vec{u}_2 = \frac{6}{5} \vec{u}_1 - \frac{12}{25} \vec{u}_2 = \begin{bmatrix} 36/5 \\ 24/5 \\ -12/5 \end{bmatrix}$

## 6.3 Example (3)

- Example 3 - Let  $W$  be a subspace spanned by the  $\vec{u}$ 's, and write  $\vec{y}$  as the sum of a vector in  $W$  and a vector orthogonal to  $W$ .

$\vec{y} = \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}, \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix}$   
 $\vec{y} = \vec{y} + \vec{z}$   
 $\hat{y} = \frac{(\vec{y} \cdot \vec{u}_1)}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \frac{(\vec{y} \cdot \vec{u}_2)}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2 = \frac{(-4+1+3)}{1+1+1} \vec{u}_1 + \frac{(-1)(-1) + (-1)(-1) + (-6)}{1+1+4} \vec{u}_2$   
 $= \frac{0}{3} \vec{u}_1 + \frac{(-2)}{6} \vec{u}_2 = -\frac{1}{3} \vec{u}_2 = \begin{bmatrix} 1/3 \\ 1/3 \\ 2/3 \end{bmatrix}$   
 $\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} -13/3 \\ 2/3 \\ 7/3 \end{bmatrix}$   
 $\vec{y} = \vec{z} + \hat{y} = \begin{bmatrix} -13/3 \\ 2/3 \\ 7/3 \end{bmatrix} + \begin{bmatrix} 1/3 \\ 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} -12/3 \\ 3/3 \\ 9/3 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}$

## 6.3 Example (4)

- Example 2 - Find the best approximation to  $\vec{z}$  by vectors of the form  $c_1 \vec{v}_1 + c_2 \vec{v}_2$ .

Same as vectors in  $\text{Span}(\vec{v}_1, \vec{v}_2)$   $\vec{z} = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix}$   
 $\hat{z} = \frac{(\vec{z} \cdot \vec{v}_1)}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 + \frac{(\vec{z} \cdot \vec{v}_2)}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2 = \frac{(4+0+1)}{4+0+1} \vec{v}_1 + \frac{(10-8+0-2)}{25+4+16+4} \vec{v}_2$   
 $= \frac{5}{5} \vec{v}_1 + \frac{0}{50} \vec{v}_2 = \vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$

## 6.4

**Gram-Schmidt Process Overview** take a given set of vectors & transform them into a set of orthogonal or orthonormal vectors

Given  $\vec{x}_1$  &  $\vec{x}_2$ , produce  $\vec{v}_1$  &  $\vec{v}_2$  where the  $\vec{v}$ 's are perp. to each other

- Let  $\vec{v}_1 = \vec{x}_1$
- Find  $\vec{v}_2$ ;  $\vec{v}_2 = \vec{x}_2 - \text{x2hat}$

$\text{x2hat} = (\vec{x}_2 \cdot \vec{v}_1) / (\vec{v}_1 \cdot \vec{v}_1) * \vec{v}_1$

## 6.4 (cont)

Orthogonal Basis  $\{v_1, v_2, \dots, v_n\}$

Orthonormal Basis  $\{v_1/\|v_1\|, v_2/\|v_2\|, \dots, v_n/\|v_n\|\}$

## 6.4 Reference (1)

Given  $\vec{x}_1, \vec{x}_2, \vec{x}_3$ , produce  $\vec{v}_1, \vec{v}_2, \vec{v}_3$

$\vec{v}_2 \perp \vec{v}_1$   
 $\vec{v}_3 \perp$  to the plane formed by  $\vec{v}_1$  and  $\vec{v}_2$

Gram-Schmidt process:

$$\vec{v}_1 = \vec{x}_1$$

$$\vec{v}_2 = \vec{x}_2 - \left(\frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right) \vec{v}_1$$

orthogonal projection of  $\vec{x}_2$  onto  $\vec{v}_1$

$$\vec{v}_3 = \vec{x}_3 - \left(\frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right) \vec{v}_1 - \left(\frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\right) \vec{v}_2$$

orthogonal projection of  $\vec{x}_3$  onto  $\vec{v}_1$  and  $\vec{v}_2$

## 6.4 Example (1)

Example - Use the Gram-Schmidt process to produce an orthogonal basis for the given subspace W:

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}$$

$$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$$\vec{v}_2 = \vec{x}_2 - \left(\frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right) \vec{v}_1 = \vec{x}_2 - \vec{v}_1 = \begin{bmatrix} 0 \\ -6 \\ 4 \end{bmatrix}$$

$$\vec{v}_3 = \vec{x}_3 - \left(\frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right) \vec{v}_1 - \left(\frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\right) \vec{v}_2 = \vec{x}_3 - \vec{v}_1 + \frac{1}{2} \vec{v}_2 = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

Orthogonal basis =  $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

## 7.1

Symmetric Matrix A square matrix where  $A^T = A$

If A is a symmetric Matrix then eigenvectors associated w/ distinct eigenvalues are orthogonal

If a matrix is symmetrical, it has an orthogonal & orthonormal basis of vectors

## 7.1 (cont)

Orthogonal matrix is a square matrix w/ orthonormal columns

(1) Matrix is square  
(2) Columns are orthogonal  
(3) Columns are unit vectors

If Matrix P has orthonormal columns  $P^T P = I$

If P is a nxn orthogonal matrix  $P^T = P^{-1}$

$A = P D P^T$

A must be symmetric, P must be normalized

## 7.1 Reference (1)

### 4. The Spectral Theorem

A set of eigenvalues of a matrix is called the spectrum of A.

- The spectral theorem for symmetric matrices - An  $n \times n$  symmetric matrix A has the following properties:
  - A has n real eigenvalues, counting multiplicities
  - The dimension of the eigenspace for each eigenvalue  $\lambda$  equals the multiplicity of  $\lambda$
  - The eigenspaces are mutually orthogonal - eigenvectors corresponding to different eigenvalues are orthogonal
  - A is orthogonally diagonalizable

Note: A symmetric matrix is always orthogonally diagonalizable but an orthogonal matrix is not necessarily orthogonally diagonalizable.

## 7.1 Example (1)

Example 2 - Determine if the following matrix is orthogonal. If it is orthogonal, find its inverse.

$$A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

Not orthogonal because the columns are not normalized

$$\|\vec{v}_1\| = \sqrt{1+4+4} = 3 \neq 1$$

## 7.1 Example (2.1)

Example 2 - Orthogonally diagonalize the following matrix, giving an orthogonal matrix P and a diagonal matrix D. Note: The eigenvalues for this matrix are 25, 3 and -50.

3x3 example

Find D:  $\lambda = 25, 3, -50$  (given)

$$D = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -50 \end{bmatrix}$$

Find P:

$\lambda = 25$

$$[(A - \lambda I) \vec{v}] = \begin{bmatrix} -2 & -36 & 0 \\ -36 & -23 & 0 \\ 0 & 0 & 3 \end{bmatrix} \vec{v}$$

normalized eigenvector  $\vec{x}_1 = \begin{bmatrix} 3/5 \\ 4/5 \\ 0 \end{bmatrix}$

$\lambda = 3$

$$[(A - \lambda I) \vec{v}] = \begin{bmatrix} -4 & -36 & 0 \\ -36 & -20 & 0 \\ 0 & 0 & 3 \end{bmatrix} \vec{v}$$

normalized eigenvector  $\vec{x}_2 = \begin{bmatrix} 3/4 \\ 0 \\ 4 \end{bmatrix}$

$\lambda = -50$

$$[(A - \lambda I) \vec{v}] = \begin{bmatrix} 48 & -36 & 0 \\ -36 & 27 & 0 \\ 0 & 0 & 53 \end{bmatrix} \vec{v}$$

normalized eigenvector  $\vec{x}_3 = \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix}$

## 7.1 Example (2.2)

$\lambda = 25$

$$[(A - \lambda I) \vec{v}] = \begin{bmatrix} -2 & -36 & 0 & 0 \\ -36 & -23 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \vec{v}$$

normalized eigenvector  $\vec{x}_1 = \begin{bmatrix} 3/5 \\ 4/5 \\ 0 \\ 0 \end{bmatrix}$

$\lambda = 3$

$$[(A - \lambda I) \vec{v}] = \begin{bmatrix} -4 & -36 & 0 & 0 \\ -36 & -20 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \vec{v}$$

normalized eigenvector  $\vec{x}_2 = \begin{bmatrix} 3/4 \\ 0 \\ 4 \\ 0 \end{bmatrix}$

$\lambda = 0$

$$[(A - \lambda I) \vec{v}] = \begin{bmatrix} 48 & -36 & 0 & 0 \\ -36 & 27 & 0 & 0 \\ 0 & 0 & 53 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \vec{v}$$

normalized eigenvector  $\vec{x}_3 = \begin{bmatrix} 1 \\ -4 \\ 0 \\ 0 \end{bmatrix}$

$P = \begin{bmatrix} 3/5 & 3/4 & 1 \\ 4/5 & 0 & -4 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

