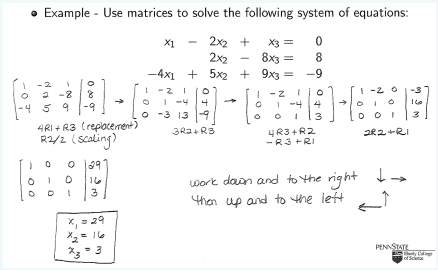
Penn State: Math 220 Cheat Sheet

by luckystarr via cheatography.com/59106/cs/15528/

1.1

A Matrix	row, columns
Coefficients Matrix	Just Left Hand Side
Augmented Matrix	Left and Right Hand Side
Solving Linear Systems	(1) Augmented Matrix (2) Row Operations (3) Solution to Linear System The RHS is the solution
One Solution	Upper triangle with Augmented Matrix
No Solution	Last row is all zeros = RHS number
Infinitely Many Solutions	Last row (including RHS) is all zeros
Inconsistent	Has No Solution

1.1 Example(1)



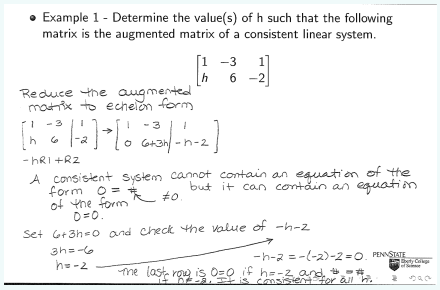
1.2

Echelon Matrix	(1) Zero Rows at the bottom (2) Leading Entries are down and to the right (3) Zeros are below each leading entry

1.2 (cont)

Reduced Echelon Matrix	(1) The leading entry of each nonzero row is 1 (2) Zeros are below AND above each 1
Pivot Position	Location of Matrix that Corresponds to a leading 1 in REF
Pivot Column	Column in Matrix that contains a pivot
To get to EF	down and right
To get to REF	up and left
Free Variables	Variables that don't correspond to pivot columns
Consistent System	Pivot in every Column

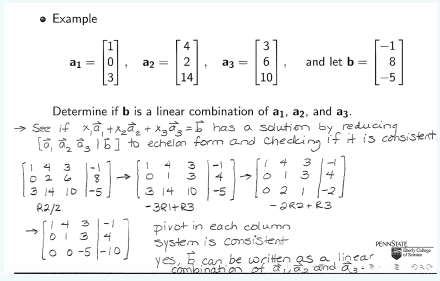
1.2 Example (1)



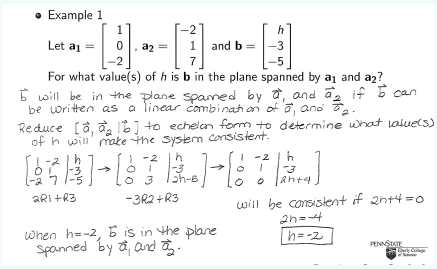
1.3

RR^2	Set of all vectors with 2 rows

1.3 Example (1)



1.3 Example (2)

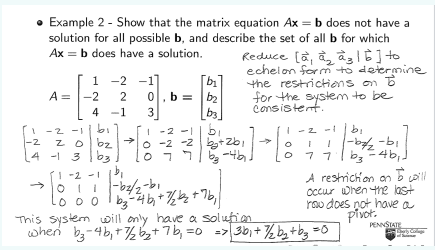


1.4

Vector Equation	x1**a1**+x2**a2**+x3**a3** =**b**
Matrix Equation	**Ax**=**b**
If A is an m x n matrix the following are all true or all false	**Ax** = **b** has a solution for every **b** in RR^m Every **b** in RR^m is a lin. combo of columns in A Columns of A span RR^m Matrix A has a pivot in every row (i.e. no row of zeros)

Anything in **Bold** means it is a vector.

1.4 Example (1)



By **luckystarr**

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Published 22nd April, 2018.

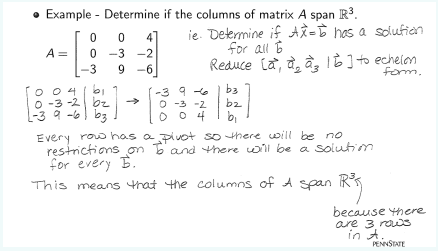
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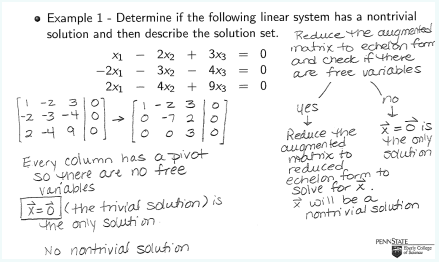
1.4 Example (2)



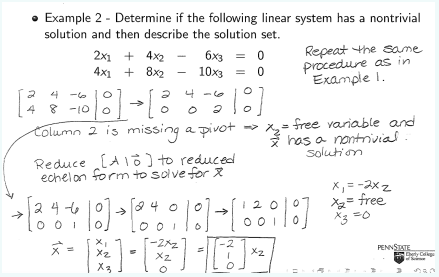
1.5

Homogeneous	**Ax = 0**
Trivial Solution	**Ax = 0** if at lease one column is missing a pivot
Determine if homogenous Linear System has a non trivial solution	(1) Write as Augmented Matrix (2) Reduce to EF (3) Determine if there are any free variables- (column w/o pivot) (4) If any free variables, than a non-trivial solution exists (5) Non-Trivial Solution can be found by further reducing to REF and solving for **x**
If **Ax = 0** has one free variable	Than **x** is a line that passes through the origin
If **Ax = 0** has two free variables	Than **x** has a plane that passes through the origin

1.5 Example (1)



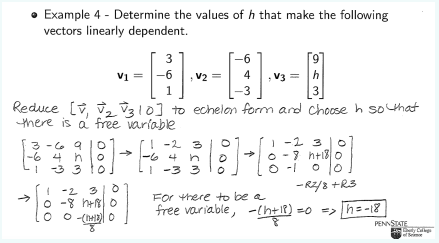
1.5 Example (2)



1.7

Linear Indepe-ndence	No free Variables, none of the vectors are multiples of each other
To check ind/dep	reduce augmented matrix to EF and see if there are free variables(ie. every column must have a pivot to be linearly independent)
To check if multiples	**u** = c * **v** find value of c, then it is a multiple therefore linearly dependent
Linearly Dependent	If there are more columns than rows

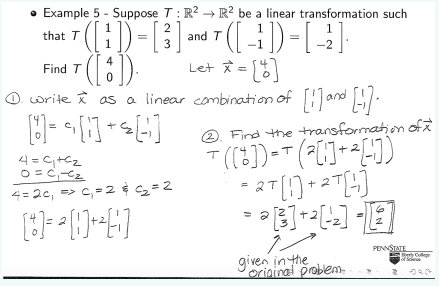
1.7 Example (1)



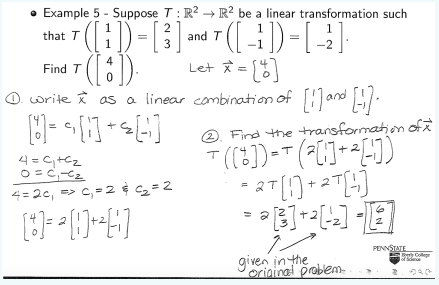
1.8

Every Matrix Transform-ation is a:	Linear Transf-ormation
T(**x**) =	A(**x**)
If A is m x n Matrix, then the properties are	(1) T(**u** + **v**) = T(**u**) + T(**v**) (2) T(c**u**) = cT(**u**) (3) T(**0**) = **0** (4) T(c**u** + d**v**) = cT(**u**) + dT(**v**)

1.8 Example (1)



1.8 Example (2)



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1.9

RR^n --> RR^m is said to be 'onto'	Equation T(**x**) =A**x**=**b** has a unique solution or more than one solution each row has a pivot
RR^n --> RR^m is said to be one-to-one	Equation T(**x**) =A**x**=**b** has a unique solution or no solution each row has a pivot

2.1

Addition of Matrices	Can Add matrices if they have same # of rows and columns (ie **A**(3x4) and **B**(3x4) so you can add them)
Multiply by Scalar	Multiply each entry by scalar
Matrix Multiplic-ation (**A** x **B**)	Must each row of **A** by each column of **B**
Powers of a Matrix	Can compute powers by if the matrix has the same number of columns as rows
Transpose of Matrix	row 1 of A becomes column 1 of A row 2 of A becomes column 2 of A

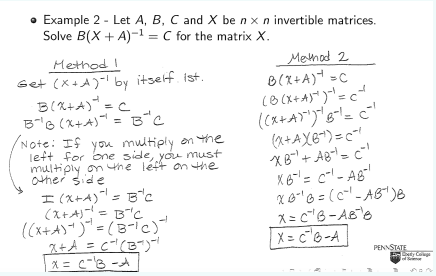
2.1 (cont)

Properties of Transpose	(1) if **A** is m x n, then $\mathbf{A}^T$ is n x m (2) $(\mathbf{A}^T)^T = \mathbf{A}$ (3) $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$ (4) $(t\mathbf{A})^T = t\mathbf{A}^T$ (5) $(\mathbf{A}\,\mathbf{B})^T = \mathbf{B}^T\mathbf{A}^T$

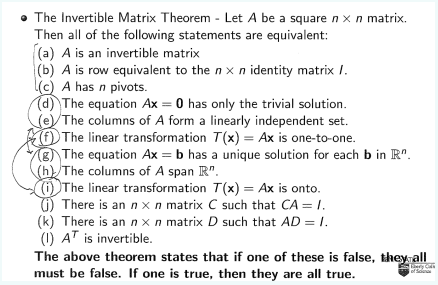
2.2

Singular matrix	A matrix that is NOT investable
Determinate of A (2 x 2) Matrix	det A = ad - bc
If A is invertable & (nxn)	There will never be no solution or infinitely many solutions to A**x** = **b**
Properties of Invertable Matricies	$(\mathbf{A}^{-1})^{-1} = \mathbf{A}$ (assuming **A** & **B** are invest-able) $(AB)^{-1} = B^{-1}A^{-1}$ $(A^T)^{-1} = (A^{-1})^T$
Finding Inverse Matrix	[A | I] --> [I | A^{-1}] *Use row operations* STOP when you get a row of Zeros, it cannot be reduced

2.2 Example (1)



2.3 Invertable Matrix Theorem

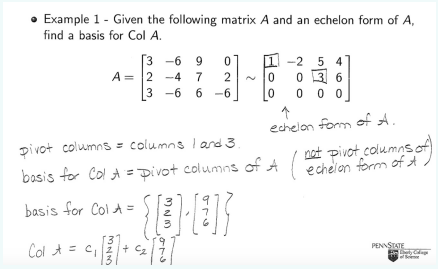


2.8

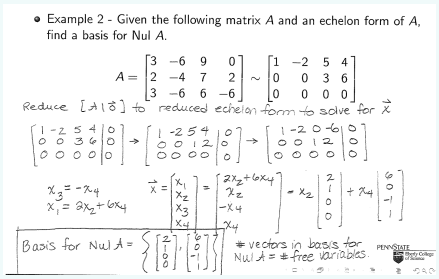
A subspace *S* of RR^n is a subspace is *S* satisfies:	(1) *S* contains zero vector (2) If **u** & **v** are in *S*, then **u** + **v** is also in *S* (3) If *r* is a real # & **u** is in *S*, then r**u** is also in *S*
Subspace RR^3	Any Plane that Passes through the origin forms a subspace RR^3 Any set that contains nonlinear terms will NOT form a subspace RR^3
Null Space (Nul A)	To determine in **u** is in the Nul(A), check if: A**u** = **0** If yes --> then **u** is in the Nullspace

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2.8 Example (1)



2.8 Example (2)

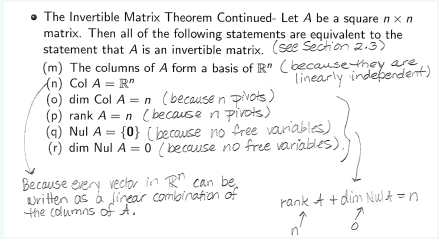


2.9

Dimension of a non-zero Subspace	# of vectors in any basis; it is the # of linearly indepe-ndent vectors
Dimension of a zero Subspace	is Zero
Dimension of a Column Space	# of pivot columns
Dimension of a Null Space	# of free variables in the solution Ax=0
Rank of a Matrix	# of pivot columns
The Rank Theorem	Matrix A has *n* columns: rank A (# pivots) + dim Nul A (# free var.) = *n*

dim = dimension; var. = variable

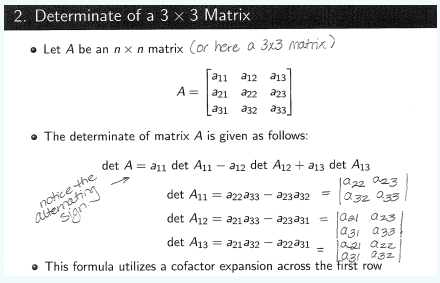
2.9 Refrence



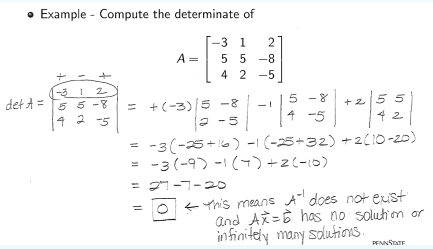
3.1

Calculating Determinant of Matrix A is another way to tell if a linear system of equations has a solution	(1) Det(A) not =0, then Ax=b has a unique solution (2) Det(A) =0, then Ax=b has no solutions or inf many
If Ax not= **0**	A^{-1} exist
If Ax = **0**	A^{-1} Does NOT exist
Cofactor Expansion	Use row/column w/ most zeros
If Matrix A has an upper or lower triangle of zeros	The det(A) is the multiplic-ation down the diagonals

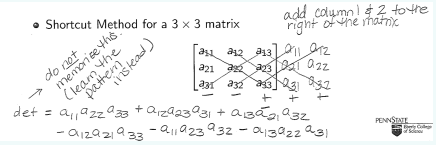
3.1 Reference (1)



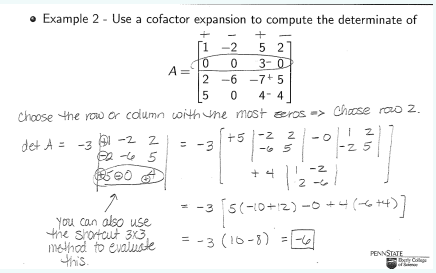
3.1 Example (1)



3.1 Reference (2)



3.1 Example (2)



3.2

Determ-inate Property 1	If a multiple of 1 row of A is added to another row to produce Matrix B, then det(B)-=det(A)
Determ-inate Property 2	If 2 rows of A are interchanged to produce B, then det(B)=-d-et(A)
Determ-inate Property 3	If one row of A is multiplied to produce B, then det(B)=k*det(A)

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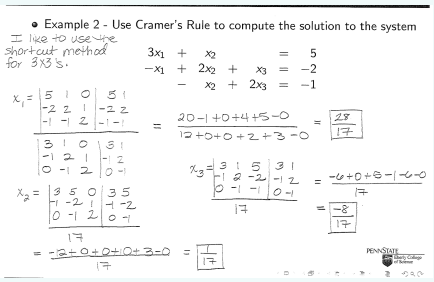
3.2 (cont)

Assuming both A & B are n x n Matrices	(1) $\det(A^T) = \det(A)$ (2) $\det(AB) = \det(A)*\det(B)$ (3) $\det(A^{-1}) = 1/\det(A)$ (4) $\det(cA) = c^n \det(A)$ (5) $\det(A^r) = (\det A)^r$

3.3 AKA Cramer's Rule

Cramer's Rule	Can be used to find the solution to a linear system of equations **Ax=b** when A is an investable square matrix
Def. of Cramer's Rule	Let A be an n x n invertible matrix. For any **b** in RR^n, the unique solution **x** of **Ax=b** has entries given by xi = detAi(**b**)/det(A) *i = 1,2,...n*
Ai(**b**)	is the matrix A w/ column i replaced w/ vector **b**

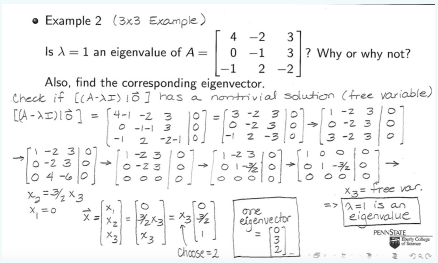
3.3 Example (1)



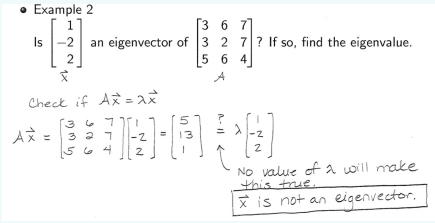
5.1

Au=λu	A is an nxn matrix. A nonzero vector **u** is an eigenvector of A if there exists such a scalar λ
To determine if λ is an eigenvalue	reduce [(A-λI)|0] to echelon form and see if it has any free variables. yes -> λ is Eigenvalue no -> λ is not eigenvalue
To determine if given vector is an eigenv-ector	**Ax=λx**
Eigenspace of A =	Nullspace of (A-λI)
Eigenvalues of triangular Matrix	entries along diagonal *you CANNOT row reduce a matrix to find its eigenv-alues

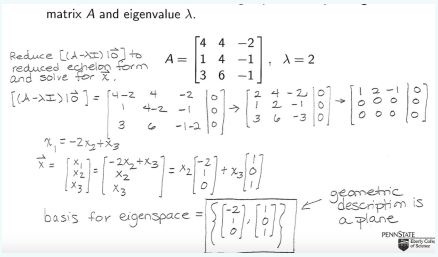
5.1 Example (1)



5.1 Example (2)



5.1 Example (3)



5.2

If λ is an eigenvalue of a Matrix A	then (A-λI)**x=0** will have a nontrivial solution
A nontrivial solution will exist	if det(A-λI)=0 (Characteristic Equation)
A is nxn Matrix. A is invertible if and only if	(1) The # 0 is NOT an λ of A (2) The det(A) is not zero
Similar Matrices	If nxn Matrices A and B are similar, then they have the same characteristic polynomial (same λ) with same multiplicities

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5.2 Example (1)

- Example 1 - Find the eigenvalues of the following matrix and state their multiplicity:

$$A = \begin{bmatrix} 9 & -2 \\ 2 & 5 \end{bmatrix}$$

$\det(A-\lambda I) = 0$

$\begin{vmatrix} 9-\lambda & -2 \\ 2 & 5-\lambda \end{vmatrix} = 0$

$(9-\lambda)(5-\lambda)+4 = 0$

$\lambda^2-14\lambda+49 = 0$

$(\lambda-7)(\lambda-7) = 0$

$\lambda = 7, 7$ or $\lambda = 7$, multiplicity 2

5.3

A matrix A written in diagonal form	$A=PDP^{-1}$
Power of Matrix	A^k = Diagonal matrix and #'s on diagonal get raised to the k
Determining if Matrix is Diagonalizable	
λ of a nxn matrix	n distinct (or real) λ then matrix is diagonalizable less than n λ, it may or may not be diagonalizable; it will be if # of linearly dependent eigenv-ectors = n
eigenv-ectors of nxn matrix	n linearly independent eigenv-ectors, then diagonalizable less than n linearly independent eigenvectors, then matrix is NOT diagonlizable
D	matrix w/ λ down diagonal

5.3 (cont)

P	columns of P have linearly n linearly independent eigenvectors
Finding *P*	solve A-λI and plug in the λ values. Reduce to EF, solve for **x**, & find eigenvector

5.3 Example (1)

- Example - Find the eigenvalues of matrix *A* and a basis for each eigenspace.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix} = \underset{P}{\begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & -3 \\ 1 & 0 & 0 \end{bmatrix}} \underset{D}{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}} \underset{P^{-1}}{\begin{bmatrix} 0 & 0 & 1 \\ -3 & 1 & 9 \\ -1 & 0 & 3 \end{bmatrix}}$$

Eigenvalues are in matrix D.

$\lambda = 3, 3, 4$

Eigenvectors are in matrix P

Basis for eigenspace associated with $\lambda = 3$ is $\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix} \right\}$

Basis for eigenspace associated with $\lambda = 4$ is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

5.3 Example (2)

- Example 3 - Find matrices *P* and *D* to diagonalize the matrix

3x3 Example with 3 eigenvalues but not enough eigenvectors

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

① Find D

$\det(A-\lambda I) = 0$

$\begin{vmatrix} 2-\lambda & 4 & 6 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 4-\lambda \end{vmatrix} = 0$

$(2-\lambda)\begin{vmatrix} 2-\lambda & 2 \\ 0 & 4-\lambda \end{vmatrix} = 0$

$(2-\lambda)(2-\lambda)(4-\lambda) = 0$

$\lambda = 2, 2, 4$

$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

② Find P

$\lambda = 2$

$[(A-\lambda I)|\vec{0}] \rightarrow \begin{bmatrix} 0 & 4 & 6 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x_2 = 0$

$x_3 = 0$

$x_1 = \text{free}$

$\vec{x} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x_1$

$\Rightarrow$ A is NOT diagonalizable

only 1 eigenvector, need 2

5.3 Example (3)

- Example 1 - Find matrices *P* and *D* to diagonalize the matrix

2x2 Example with 2 eigenvalues and 2 eigenvectors.

$$A = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} \qquad P = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$$

① Find D

$\det(A-\lambda I) = 0$

$\begin{vmatrix} 3-\lambda & 1 \\ -2 & -\lambda \end{vmatrix} = 0$

$(3-\lambda)(-\lambda)+2 = 0$

$\lambda^2-3\lambda+2 = 0$

$(\lambda-2)(\lambda-1) = 0$

$\lambda = 1, 2$

$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

② Find P

$\lambda = 1$

$[(A-\lambda I)|\vec{0}] \rightarrow \begin{bmatrix} 2 & 1 & 0 \\ -2 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 = -\frac{1}{2}x_2$

$\vec{x} = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} x_2 \overset{x_2=2}{\Rightarrow}$ eigenvector $= \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\lambda = 2$

$[(A-\lambda I)|\vec{0}] \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ -2 & -2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 = -x_2$

$\vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} x_2$, $x_2 = 1$ eigenvector $= \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

6.1

Length of vector **x**	$\|\|\mathbf{x}\|\| = \sqrt{x1^2+x2^2}$
Length fo vector **x** in RR^2	$\|\|\mathbf{x}\|\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$
The Unit Vector	$\mathbf{u} = \mathbf{v}/\|\|\mathbf{v}\|\|$
Two vectors **u** & **v** in RR^n, the distance between **u** & **v**	$\|\|\mathbf{u} - \mathbf{v}\|\|$
Two vectors **u** & **v** are orthogonal if and only if	$\|\|u+v\|\|^2 = \|\|u\|\|^2 + \|\|v\|\|^2$ $\mathbf{u} \cdot \mathbf{v} = 0$

6.2

The distance from **y** to the line through **u** & the origin	$\|\|z\|\| = \|\|y - \text{y-hat}\|\|$

6.2 Example (1)

- Example 1 - Determine if $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ is an orthogonal basis for $\mathbb{R}^3$.

$$\mathbf{u_1} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \quad \mathbf{u_2} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{u_3} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

Check if $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ are orthogonal and nonzero.

→ They are all nonzero.

→ Check orthogonality

$\vec{v}_1 \cdot \vec{v}_2 = 6-6+0 = 0$ ✓

$\vec{v}_1 \cdot \vec{v}_3 = 3-3+0 = 0$ ✓

$\vec{v}_2 \cdot \vec{v}_3 = 2+2-4 = 0$ ✓

yes, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ forms an orthogonal basis for $\mathbb{R}^3$ because each $\vec{v}$ has 3 rows.

6.2 Example (2)

- Example 2 - Write **x** as a linear combination of $\mathbf{u_1}, \mathbf{u_2}$, and $\mathbf{u_3}$.

$$\mathbf{x} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}, \quad \mathbf{u_1} \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \quad \mathbf{u_2} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{u_3} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$

Note that the $\vec{v}$'s are orthogonal.

$c_1 = \frac{\vec{x} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} = \frac{15+9+0}{9+9} = \frac{24}{18} = \frac{4}{3}$

$c_2 = \frac{\vec{x} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} = \frac{10-6-1}{4+4+1} = \frac{3}{9} = \frac{1}{3}$

$c_3 = \frac{\vec{x} \cdot \vec{v}_3}{\vec{v}_3 \cdot \vec{v}_3} = \frac{5-3+4}{1+1+16} = \frac{6}{18} = \frac{1}{3}$

$\vec{x} = \frac{4}{3}\vec{x}_1 + \frac{1}{3}\vec{x}_2 + \frac{1}{3}\vec{x}_3$

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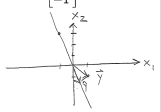
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6.2 Example (3)

- Example 1 - Compute the orthogonal projection of $\begin{bmatrix}1\\-1\end{bmatrix}$ onto the line through $\begin{bmatrix}-1\\3\end{bmatrix}$ and the origin.



$\vec{y} = \begin{bmatrix}1\\-1\end{bmatrix}$ = vector being projected

$\vec{u} = \begin{bmatrix}-1\\3\end{bmatrix}$ = vector that the line passes through

orthogonal projection $= \hat{y} = \left(\frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)\vec{u}$

$= \frac{(-1-3)}{(-1)^2+(3)^2}\vec{u}$

$= -\frac{4}{10}\vec{u} = -\frac{2}{5}\vec{u} = \begin{bmatrix}2/5\\-6/5\end{bmatrix}$

6.2 Example (4)

- Example 2 - Let $\mathbf{y} = \begin{bmatrix}2\\6\end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix}7\\1\end{bmatrix}$. Write $\mathbf{y}$ as the sum of a vector in Span$\{\mathbf{u}\}$ and a vector orthogonal to $\mathbf{u}$.

$= \alpha\vec{u} = \hat{y}$ $= \vec{z}$

$\vec{y} = \hat{y} + \vec{z}$

$\hat{y} = \left(\frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)\vec{u} = \left(\frac{14+6}{49+1}\right)\vec{u} = \frac{2}{5}\vec{u} = \begin{bmatrix}14/5\\2/5\end{bmatrix}$

$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix}2\\6\end{bmatrix} - \begin{bmatrix}14/5\\2/5\end{bmatrix} = \begin{bmatrix}-4/5\\28/5\end{bmatrix}$

$\vec{y} = \begin{bmatrix}14/5\\2/5\end{bmatrix} + \begin{bmatrix}-4/5\\28/5\end{bmatrix}$

6.2 Example (5)

- Example 3 - Let $\mathbf{y} = \begin{bmatrix}-3\\9\end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix}1\\2\end{bmatrix}$. Compute the distance from $\mathbf{y}$ to the line through $\mathbf{u}$ and the origin.

distance $= \|\vec{y} - \hat{y}\|$

① Find $\hat{y}$

$\hat{y} = \left(\frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)\vec{u} = \left(\frac{-3+18}{1+4}\right)\vec{u} = 3\vec{u} = \begin{bmatrix}3\\6\end{bmatrix}$

② Find $\vec{y} - \hat{y}$

$\vec{y} - \hat{y} = \begin{bmatrix}-3\\9\end{bmatrix} - \begin{bmatrix}3\\6\end{bmatrix} = \begin{bmatrix}-6\\3\end{bmatrix}$

③ Find $\|\vec{y} - \hat{y}\|$

$\|\vec{y} - \hat{y}\| = \sqrt{(-6)^2+(3)^2} = \sqrt{36+9} = \sqrt{45} = 3\sqrt{5}$

6.2 Example (6)

- Example 1 - Determine if the following set of vectors is orthonormal. If it is only orthogonal, normalize the vectors to produce an orthonormal set.

$\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\0\end{bmatrix}$ (orthogonal and normalized)

$\vec{v}_1$ $\vec{v}_2$

① Check if they are orthogonal

$\vec{v}_1 \cdot \vec{v}_2 = 0-1+0 \neq 0$ => not orthogonal => not orthonormal

② No need to check if they are normalized since they are not orthogonal

6.2 Example (7)

- Example 2- Determine if the following set of vectors is orthonormal. If it is only orthogonal, normalize the vectors to produce an orthonormal set.

$\begin{bmatrix}-2/3\\1/3\\2/3\end{bmatrix}, \begin{bmatrix}1/3\\2/3\\0\end{bmatrix}$ (orthogonal and normalized)

$\vec{v}_1$ $\vec{v}_2$

① Check if they are orthogonal

$\vec{v}_1 \cdot \vec{v}_2 = -2/9+2/9+0 = 0$ ✓

② Check if they are normalized

$\|\vec{v}_1\| = \sqrt{4/9+1/9+4/9} = 1$ ✓

$\|\vec{v}_2\| = \sqrt{1/9+4/9} = \frac{\sqrt{5}}{3} \neq 1$

$\frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{\vec{v}_2}{\sqrt{5}/3} = \frac{3}{\sqrt{5}}\vec{v}_2$

orthonormal set $= \left\{\vec{v}_1, \frac{\vec{v}_2}{\|\vec{v}_2\|}\right\} = \left\{\begin{bmatrix}-2/3\\1/3\\2/3\end{bmatrix}, \begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\\0\end{bmatrix}\right\}$

6.2 Reference (1)

- We end with the formula for the orthogonal projection of $\mathbf{y}$ onto L and the component of $\mathbf{y}$ orthogonal to L:

memorize this!

$\hat{\mathbf{y}} = \left(\frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}$ and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$ (or onto a line through $\vec{0}$ and the origin)

- Note that $\hat{\mathbf{y}}$ is in Span$\{\mathbf{u}\}$ (because $\hat{y} = \alpha\vec{u}$ a linear combination of $\vec{u}$)

6.2 Reference (2)

$\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p\} \rightarrow \left\{\frac{\vec{v}_1}{\|\vec{v}_1\|}, \frac{\vec{v}_2}{\|\vec{v}_2\|}, \cdots, \frac{\vec{v}_p}{\|\vec{v}_p\|}\right\}$

orthogonal set → orthonormal set

6.3 Example (1.1)

- Example - Assume that $\{\mathbf{u}_1, \cdots, \mathbf{u}_4\}$ is an orthogonal basis for $\mathbb{R}^4$. Write $\mathbf{x}$ as the sum of two vectors, one in Span $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and one in Span$\{\mathbf{u}_4\}$.

$\vec{v}_2$ $\vec{v}_1$

$\mathbf{x} = \begin{bmatrix}4\\5\\-3\\3\end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix}-1\\1\\1\\-2\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}-2\\1\\-1\\1\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}1\\1\\-2\\-1\end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix}1\\2\\1\\1\end{bmatrix}$

$\vec{x} = \vec{v}_1 + \vec{v}_2$

$\vec{v}_1$ = vector in Span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$

$\vec{v}_2$ = vector in Span $\{\vec{v}_4\} = c_4\vec{u}_4$

Find c_1, c_2, c_3 and c_4 by using the fact that the $\vec{v}$'s are orthogonal.

6.3 Example (1.2)

Find c_4 1st (because it's easier!)

$\vec{x} = \vec{v}_1 + \vec{v}_2 = c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 + c_4\vec{u}_4$

$\vec{x}\cdot\vec{u}_4 = c_1\vec{u}_1\cdot\vec{u}_4 + c_2\vec{u}_2\cdot\vec{u}_4 + c_3\vec{u}_3\cdot\vec{u}_4 + c_4\vec{u}_4\cdot\vec{u}_4$ (the first three terms are 0)

$c_4 = \frac{\vec{x}\cdot\vec{u}_4}{\vec{u}_4\cdot\vec{u}_4} = \frac{4+10-3+3}{1+4+1+1} = \frac{14}{7} = 2$ $\Rightarrow \vec{v}_2 = c_4\vec{u}_4 = \begin{bmatrix}2\\4\\2\\2\end{bmatrix}$

To find $\vec{v}_1$, use the fact that

$\vec{x} = \vec{v}_1 + \vec{v}_2$

$\vec{v}_1 = \vec{x} - \vec{v}_2 = \begin{bmatrix}4\\5\\-3\\3\end{bmatrix} - \begin{bmatrix}2\\4\\2\\2\end{bmatrix} = \begin{bmatrix}2\\1\\-5\\1\end{bmatrix}$

$\vec{x} = \vec{v}_1 + \vec{v}_2 = \begin{bmatrix}2\\1\\-5\\1\end{bmatrix} + \begin{bmatrix}2\\4\\2\\2\end{bmatrix}$

6.3 Example (2)

- Example 1 - Verify that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set, and then find the orthogonal projection of $\mathbf{y}$ onto Span$\{\mathbf{u}_1, \mathbf{u}_2\}$.

$\mathbf{y} = \begin{bmatrix}6\\3\\-2\end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix}3\\4\\0\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}-4\\3\\0\end{bmatrix}$

$\vec{v}_1 \cdot \vec{v}_2 = -12+12+0 = 0$ ✓ $\Rightarrow \vec{v}_1 \perp \vec{v}_2$

Orthogonal projection of $\vec{y}$ onto Span $\{\vec{v}_1, \vec{v}_2\}$

$= \hat{y} = \left(\frac{\vec{y}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1 + \left(\frac{\vec{y}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\right)\vec{v}_2$ $= \frac{6}{5}\begin{bmatrix}3\\4\\0\end{bmatrix} - \frac{3}{5}\begin{bmatrix}-4\\3\\0\end{bmatrix}$

$= \left(\frac{18+12+0}{9+16}\right)\vec{v}_1 + \left(\frac{-24+9+0}{16+9}\right)\vec{v}_2$ $= \begin{bmatrix}6\\3\\0\end{bmatrix}$

$= \frac{30}{25}\vec{v}_1 - \frac{15}{25}\vec{v}_2 = \frac{6}{5}\vec{v}_1 - \frac{3}{5}\vec{v}_2$

6.3 Example (3)

- Example 3 - Let W be a subspace spanned by the $\mathbf{u}$'s, and write $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W.

$\hat{y}$ $\vec{z}$

$\mathbf{y} = \begin{bmatrix}-1\\4\\3\end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}-1\\3\\-2\end{bmatrix}$

$\vec{y} = \hat{y} + \vec{z}$

$\hat{y} = \left(\frac{\vec{y}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1 + \left(\frac{\vec{y}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\right)\vec{v}_2$

$\hat{y} = \left(\frac{-1+4+3}{1+1+1}\right)\vec{v}_1 + \left(\frac{1+12-6}{1+9+4}\right)\vec{v}_2$

$\hat{y} = 2\vec{v}_1 + \frac{1}{2}\vec{v}_2 = 2\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}-1\\3\\-2\end{bmatrix} = \begin{bmatrix}3/2\\7/2\\1\end{bmatrix}$

$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix}-1\\4\\3\end{bmatrix} - \begin{bmatrix}3/2\\7/2\\1\end{bmatrix} = \begin{bmatrix}-5/2\\1/2\\2\end{bmatrix}$

$\vec{y} = \begin{bmatrix}3/2\\7/2\\1\end{bmatrix} + \begin{bmatrix}-5/2\\1/2\\2\end{bmatrix}$

6.3 Example (4)

- Example 2 - Find the best approximation to $\mathbf{z}$ by vectors of the form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$.

$\rightarrow = \hat{z}$ = closest point to $\vec{z}$

Same as vectors in Span $\{\vec{v}_1, \vec{v}_2\}$

$\mathbf{z} = \begin{bmatrix}2\\4\\0\\-1\end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix}2\\0\\-1\\-3\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}5\\-2\\4\\2\end{bmatrix}$

$\hat{z} = \left(\frac{\vec{z}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1 + \left(\frac{\vec{z}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\right)\vec{v}_2$

$\hat{z} = \left(\frac{4+0+0+3}{4+0+1+9}\right)\vec{v}_1 + \left(\frac{10-8+0-2}{25+4+16+4}\right)\vec{v}_2$

$\hat{z} = \frac{1}{2}\vec{v}_1 + 0\vec{v}_2 = \begin{bmatrix}1\\0\\-1/2\\-3/2\end{bmatrix}$

6.4

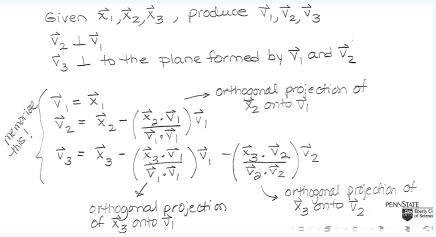
Gram- Schmidt Process Overview	take a given set of vectors & transform them into a set of orthogonal or orthonormal vectors
Given **x1** & **x2**, produce **v1** & **v2** where the v's are perp. to each other	(1) Let **v1=x1** (2) Find **v2**; **v2=x2 - x2hat**
x2 hat	(x2•v1)/(v1•v1) * v1

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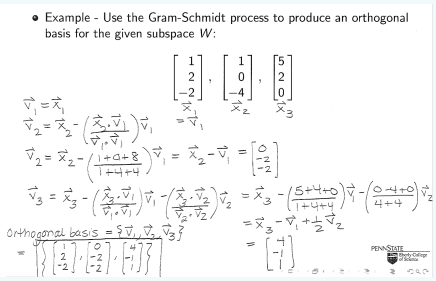
6.4 (cont)

Orthogonal Basis	{**v1**,**v2**,...,**vn**}
Orthonormal Basis	{**v1**/||**v1**||, **v2**/ ||**v2**||,..., **vn**/||**vn**||}

6.4 Reference (1)



6.4 Example (1)



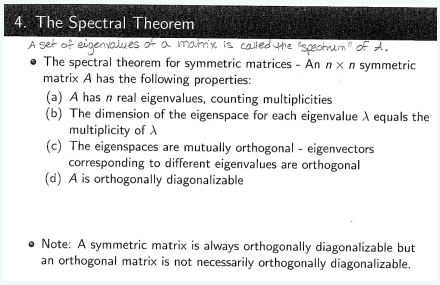
7.1

Symmetric Matrix	A square matrix where $A^T=A$
If A is a symmetric Matrix	then eigenvectors associated w/ distinct eigenvalues are orthogonal If a matrix is symmetrical, it has an orthogonal & orthonormal basis of vectors

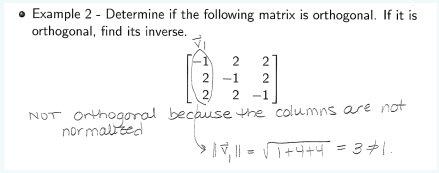
7.1 (cont)

Orthogonal matrix is a square matrix w/ orthonormal columns	(1) Matrix is square (2) Columns are orthogonal (3) Columns are unit vectors
If Matrix P has orthonormal columns	$P^TP=I$
If P is a nxn orthogonal matrix	$P^T=P^{-1}$
$A=PDP^T$	A must be symmetric, P must be normalized

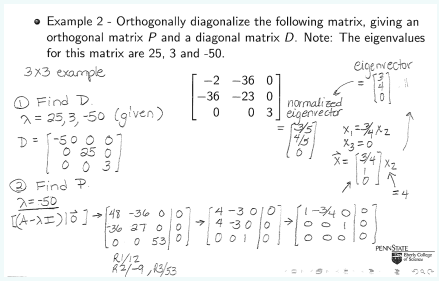
7.1 Reference (1)



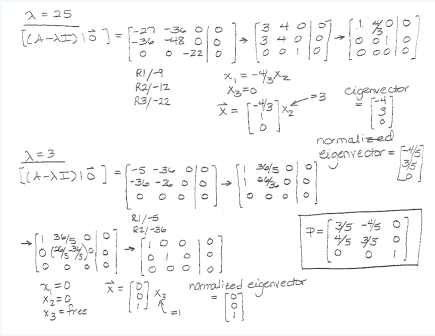
7.1 Example (1)



7.1 Example (2.1)



7.1 Example (2.2)



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