Appendix B+ - Measure & Probability Theory Cheat Sheet

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σ-algebras & Borel sets

$\mathscr{A} \subseteq 2^{\Omega}$ is a σ-algebra	if $\Omega \in \mathscr{A}$, $A \in \mathscr{A} \Rightarrow A^c \in$ □ □, and $A_i \in \mathscr{A} \Rightarrow \cup_{i \in N} A_i \in$ □ □
$\sigma(\mathscr{E})$	intersection of all σ-algebras containing $\mathscr{E}$
$\mathscr{B}(V)$	σ(open subsets of V)

Measures

Measure	$\mu(\emptyset)=0$ and $\mu(\cup A_i)=\Sigma\, \mu(A_i)$ for disjoint A_i
Probability	$P(\Omega)=1$
Product	$(\otimes_j \mu_j)(A_1 \times \ldots \times A_n)=\prod_j \mu_j(A_j)$
Lebesgue	$\lambda^d(\prod_j (a_j,b_j)) = \prod_j (b_j-a_j)$

Measurability

Measurable	$f^{1}(A_2) \in \mathscr{A}_1$ for all $A_2 \in \mathscr{A}_2$
Check on generators	$\mathscr{A}_2=\sigma(\mathscr{E})$, enough to check $f^{1}(E) \in \mathscr{A}_1$ for $E \in \mathscr{E}$
Strongly measurable	f_n simple, $f_n \to f$ pointwise or μ-a.e.
Pettis	f strongly measurable ⇔ f separably valued and ⟨f,v′⟩ measurable ∀v′ ∈ V

Bochner integration, change of variables

μ-simple	$f = \Sigma_{j=1}^{n} 1_{A\_j}\, v_j$, $\mu(A_j) < \infty$
μ-simple f	$\int f\, d\mu = \Sigma\, \mu(A_j)\, v_j$
Bochner integrable	$f_n \to f$ μ-a.e. and $\int \|f-f_n\|\, d\mu \to 0$
Bochner criterion	f strongly μ-measurable, $\int \|f\|\, d\mu < \infty$
Norm bound	$\|\int f\, d\mu\| \le \int \|f\|\, d\mu$

Bochner integration, change of variables (cont)

Duality	$\langle \int f\, d\mu, v' \rangle = \int \langle f,v' \rangle\, d\mu$
L^p	μ-a.e. equivalence classes of strongly μ-measurable functions with finite L^p norm
dν/dμ	density of ν w.r.t μ, ν ≪ μ
Pushforward	$T_{\#}\, \mu(A) = \mu(T^{-1}(A))$ $\int f\, d(T_{\#}\mu) = \int f \circ T\, d\mu$

Banach-valued RVs

Random variable	$X:(\Omega,\mathscr{A}) \to (V,B(V))$ measurable
$P_X = X_{\#}P$, so $P[X \in B]=P_X(B)$	
$\sigma(X) = \{X^{-1}(B): B \in \mathscr{B}(V)\}$	
$E[X] = \int_{\Omega} X(\omega)dP(\omega)$, if $\int \|X\|\, dP < \infty$	
$E[\varphi(X)] = \int_{\Omega} \varphi(X(\omega))dP(\omega) = \int_V \varphi(v)dP_X(v)$	

Conditional probability & independence

1st Def.	$P[A\|B] = P[A \cap B]/P[B]$, $P[B]>0$
A,B independent ⇔ $P[A \cap B]=P[A]P[B]$	
X_i independent ⇔ $\sigma(X_i)$ independent	
If $X_1,\ldots,X_n$ independent and integrable: $E[\prod X_j]=\prod E[X_j]$	

Conditional expectation

For simple Y $= \Sigma\, 1_{A\_j}\, y_j$	$E[X\|Y](\omega) = 1/P[A_j]\int_{A\_j} X\, dP$ for $\omega \in A_j$
1st Def.	$Z=E[X\|\mathscr{F}]$ iff Z is $\mathscr{F}$-meas. and $\int_B Z\, dP=\int_B X\, dP$ $\forall B \in \mathscr{F}$
2nd Def.	$E[X\|Y] := E[X\|\sigma(Y)]$ $P[A\|Y] := E[1_A\|\sigma(Y)]$

Regular conditional distribution

Goal:	define $P[X \in B \mid Y=y]$ even when $P[Y=y]=0$
Markov kernel	$\kappa_{X\|F}: \Omega \times \mathscr{B}(V) \to [0,1]$
reg. cond. distr. of X given $\mathscr{F}$	$P[A \cap \{X \in B\}] = \int_A \kappa_{X\|F}(\omega,B)dP(\omega)$, $\forall A \in \mathscr{F}$, $B \in \mathscr{B}(V)$
Doob-Dynkin	κ σ(Y)-measurable ⇔ κ = τ∘Y for measurable τ
reg. cond. distr. of X given Y	$P[X \in B \mid Y=y] := \tau_{X\|Y}(y,B) := \kappa_{X\|\sigma(Y)}(Y^{-1}(y),B)$

Conditional densities

$f_Y(y) = \int_{R^m} f_{X,Y}(x,y)\, dx$	
$f_{X\|Y}(x\|y) = f_{X,Y}(x,y)/f_Y(y)$, for $f_Y(y)>0$	
$P[X \in B \mid Y=y] = \int_B f_{X\|Y}(x\|y)\, dx$	

Gaussians

$X \sim N(\mu,\sigma^2)$:	$f_X(x) = 1/\sqrt{(2\pi\sigma^2)}\, \exp(-(x-\mu)^2/(2\sigma^2))$
$X \sim N(\mu,\Sigma)$:	$f_X(x) \propto \det(\Sigma)^{-1/2} \exp(-1/2 (x-\mu)^T \Sigma^{-1} (x-\mu))$
Gaussian facts	marginals Gaussian; conditionals Gaussian; jointly Gaussian + uncorrelated ⇔ independent

Distances & Divergences

$D_{TV}(P,Q) = \sup_{A \in \mathscr{A}} \|P(A)-Q(A)\|$	
$D_H(P,Q) = (1/\sqrt{2})(\int (\sqrt{(dP/d\mu)}-\sqrt{(dQ/d\mu)})^2\, d\mu)^{1/2}$	
$D_{KL}(P\|\|Q) = \int \log(dP/dQ)dP$ if $P \ll Q$; ∞ otherwise	
Expectation bound	$\|E_P[f] - E_Q[f]\| \le 2D_H(P,Q)\, (E_P\|f\|^2+E_Q\|f\|^2)^{1/2}$