Appendix B+ - Measure & Probability Theory Cheat Sheet

by LuciaZhang via cheatography.com/136048/cs/48168/

σ-algebras & Borel sets

$A \subseteq 2^{\Omega}$ is a σ-algebra	if $\Omega \in A$, $A \in A \Rightarrow A^c \in A$, and $A_i \in A \Rightarrow \cup_{i\in N} A_i \in A$
σ(E)	intersection of all σ-algebras containing E
$\mathscr{B}(V)$	σ(open subsets of V)

Measures

Measure	$\mu(\varnothing)=0$ and $\mu(\cup A_i)=\Sigma\, \mu(A_i)$ for disjoint A_i
Probability	$P(\Omega)=1$
Product	$(\otimes_j\mu_j)(A_1 \times ... \times A_n)=\prod_j \mu_j(A_j)$
Lebesgue	$\lambda^d(\prod_j (a_j,b_j)) = \prod_j (b_j-a_j)$

Measurability

Measurable	$f^{-1}(A_2) \in \mathscr{A}_1$ for all $A_2 \in \mathscr{A}_2$
Check on generators	$\mathscr{A}_2=\sigma(\mathscr{E})$, enough to check $f^{-1}(E) \in A_1$ for $E \in \mathscr{E}$
Strongly measurable	f_n simple, $f_n \to f$ pointwise or μ-a.e.
Pettis	f strongly measurable ⇔ f separably valued and ⟨f,v′⟩ measurable for every v′ ∈ V

Bochner integration, change of variables

μ-simple	$f = \Sigma_{j=1}^{n} 1_{A\_j}\, v_j$, $\mu(A_j) < \infty$
	$\int f\, d\mu = \Sigma\, \mu(A_j)\, v_j$ for simple f
Bochner definition	$f\_n \to f$ μ-a.e. and $\int \|\|f-f\_n\|\|\, d\mu \to 0$
Bochner criterion	f strongly μ-measurable, $\int \|\|f\|\|\, d\mu < \infty$

Bochner integration, change of variables (cont)

Norm bound	$\|\|\int f\, d\mu\|\| \le \int \|\|f\|\|\, d\mu$
Duality	$\langle \int f\, d\mu, v' \rangle = \int \langle f,v' \rangle\, d\mu$
L^p	μ-a.e. equivalence classes of strongly μ-measurable functions with finite L^p norm
dν/dμ	density of ν w.r.t μ, ν ≪ μ
Pushforward	$T_{\#}\, \mu(A) = \mu(T^{-1}(A))$ $\int f\, d(T_{\#}\mu) = \int f \circ T\, d\mu$

Banach-valued RVs

Random variable	X:(Ω,A)→(V,B(V)) measurable

$P_X = X_{\#}P$, so $P[X \in B]=P_X(B)$

$\sigma(X) = \{X^{-1}(B): B \in \mathscr{B}(V)\}$

$E[X] = \int_{\Omega} X(\omega)dP(\omega)$, if $\int \|\|X\|\|\, dP < \infty$

$E[\varphi(X)] = \int_{\Omega} \varphi(X(\omega))dP(\omega) = \int_V \varphi(v)dP_X(v)$

Conditional probability & independence

$P[A|B] = P[A \cap B]/P[B]$, $P[B]>0$

A,B independent ⇔ $P[A\cap B]=P[A]P[B]$

X_i independent ⇔ $\sigma(X_i)$ independent

If $X_1,...,X_n$ independent and integrable: $E[\prod X_i]=\prod E[X_i]$

Conditional expectation

For simple $Y = \Sigma\, 1_{A\_j}\, y_j$	$E[X\|Y](\omega) = 1/P[A_j] \int_{A\_j} X\, dP$ for $\omega \in A_j$
$Z=E[X\|\mathscr{F}]$	iff Z is $\mathscr{F}$-measurable and $\int_B Z\, dP=\int_B X\, dP$ for all B ∈ F

$E[X|Y] := E[X|\sigma(Y)]$, $P[A|Y] := E[1_A|\sigma(Y)]$

Regular conditional distributions & Doob-Dynkin

Goal:	define $P[X \in B \mid Y=y]$ even when $P[Y=y]=0$

$\kappa_{X\|F}: \Omega \times \mathscr{B}(V) \to [0,1]$

For $A\in\mathscr{F}$, $B\in\mathscr{B}(V)$: $P[A\cap\{X\in B\}] = \int_A \kappa_{X\|F}(\omega,B)dP(\omega)$

Doob-Dynkin	κ σ(Y)-measurable ⇔ κ = τ∘Y for measurable τ

$\tau_{X\|Y}(y,B) := P[X\in B \mid Y=y]$

Conditional densities

$f_Y(y) = \int_{R^m} f_{X,Y}(x,y)\, dx$

$f_{X\|Y}(x\|y) = f_{X,Y}(x,y)/f_Y(y)$, for $f_Y(y)>0$

$P[X \in B \mid Y=y] = \int_B f_{X\|Y}(x\|y)\, dx$

Gaussians

$X\sim N(\mu,\sigma^2)$:	$f_X(x) = 1/\sqrt{(2\pi\sigma^2)}\, \exp(-(x-\mu)^2/(2\sigma^2))$
$X\sim N(\mu,\Sigma)$:	$f_X(x) \propto \det(\Sigma)^{-1/2} \exp(-1/2 (x-\mu)^T\Sigma^{-1}(x-\mu))$
Gaussian facts	marginals Gaussian; conditionals Gaussian; jointly Gaussian + uncorrelated ⇔ independent

Distances & Divergences

$D_{TV}(P,Q) = \sup_{A\in\mathscr{A}}\|P(A)-Q(A)\|$

$D_H(P,Q) = (1/\sqrt{2})(\int(\sqrt{(dP/d\mu)}-\sqrt{(dQ/d\mu)})^2\, d\mu)^{1/2}$

$D_{KL}(P\|\|Q) = \int \log(dP/dQ)dP$ if P≪Q; ∞ otherwise

Expectation bound	$\|\|E_P[f] - E_Q[f]\|\| \le 2D_H(P,Q)\, (E_P\|\|f\|\|^2+E_Q\|\|f\|\|^2)^{1/2}$