

Normed spaces and Banach spaces

Strong convergence $\|x_n - x\|_X \rightarrow 0$

Continuous embedding $X \subset Y$ and $\|x\|_Y \leq C\|x\|_X$

Banach space = complete normed space

Closure = add all limits of convergent sequences from the set

Bounded linear operators

$D(A) \subset X$, $N(A) \subset X$, $R(A) \subset Y$

$A \in L(X, Y) \iff A$ linear and bounded

Boundedness $\|Ax\|_Y \leq C\|x\|_X$

Operator norm $\|A\| = \sup_{x \neq 0} \frac{\|Ax\|_Y}{\|x\|_X}$

Inverse bounded $\iff c\|x\|_X \leq \|Ax\|_Y$ ($c > 0$)

Convergence, Banach-Steinhaus

Pointwise $\forall x, A_n x \rightarrow Ax \in Y$

Uniform $\|A_n - A\|_{L(X, Y)} \rightarrow 0$

Banach-Steinhaus pointwise bounded family $\implies \sup_j \|A_j\| < \infty$

Hilbert spaces, weak convergence

Hilbert space = Banach space + scalar product

Weak convergence $x_n \rightharpoonup x \iff \langle x_n, z \rangle \rightarrow \langle x, z \rangle, \forall z \in X$

Strong \implies weak. In infinite dims: weak $\not\Rightarrow$ strong

Weak + convergence of norms \implies strong

Riesz representation, adjoint operators

Riesz $\lambda(x) = \langle z_\lambda, x \rangle$

Adjoint $\langle A^* y, x \rangle_X = \langle y, Ax \rangle_Y$

Matrix case $A^* = A^T$ over \mathbb{R}

Properties $(A^*)^* = A$, $\|A^*\| = \|A\|$, $\|A^* A\| = \|A\|^2$

Orthogonality and projections

$U^\perp = \{x \in X : \langle x, u \rangle = 0 \forall u \in U\}$

U^\perp is always closed

If U closed: $X = U \oplus U^\perp$

If U not closed: $(U^\perp)^\perp = \text{cl}(U)$

Projection $z = P_U x \iff z \in U$ and $x - z \in U^\perp$

Range/kernel identities

$R(A)^\perp = N(A^*)$ $N(A^*)^\perp = \text{cl}(R(A))$

$R(A^*)^\perp = N(A)$ $N(A)^\perp = \text{cl}(R(A^*))$

Kernel closed, range may be non-closed

Orthonormal systems and bases

Orthonormal $\langle u_i, u_j \rangle = \delta_{ij}$

Bessel $\sum_j |\langle x, u_j \rangle|^2 \leq \|x\|^2$

ONB expansion $x = \sum_j \langle x, u_j \rangle u_j$

Projection $P_U x = \sum_j \langle x, u_j \rangle u_j$

Compact operators

Compact bounded sequence $x_n \rightarrow Ax_n$ has a convergent subsequence

Equivalent weak convergence in $X \rightarrow$ strong convergence after A in Y

Finite-dimensional range \implies compact

I_d compact $\iff \dim(X) < \infty$

$A, S \in L(X, Y)$, A or S compact $\implies A \circ S$ compact

A compact $\iff A^*$ compact

Example: integral operators

$Kx(t) = \int_0^1 k(s, t)x(s)ds$

Approximate k by piecewise-constant k_n

K_n finite rank $\implies K_n$ compact;

$K_n \rightarrow K$ in operator norm $\implies K$ compact.

Self-adjoint $\iff k(s, t) = k(t, s)$ a.e. (real case)

Spectral theorem (for compact, $K=K^*$)

$Kx = \sum_n \lambda_n \langle x, u_n \rangle u_n$, with $\lambda_n \rightarrow 0$

$Ku_n = \lambda_n u_n$, $\{u_n\}$ ONB for $\text{cl}(R(K))$

$\|K\| = |\lambda_1|$ when ordered by $|\lambda_1| \geq |\lambda_2| \geq \dots \geq 0$

Inverse danger: divide by small λ_n

