6A The Language of Polynomials

Example 1

Let $P(x) = x^4 - 3x^3 - 2$. Find:

a $P(1)$ **b** $P(-1)$ **c** $P(2)$ **d** $P(-2)$

Solution

a $P(1) = 1^4 - 3 \times 1^3 - 2 = 1 - 3 - 2 = -4$

b $P(-1) = (-1)^4 - 3 \times (-1)^3 - 2 = 1 + 3 - 2 = 2$

c $P(2) = 2^4 - 3 \times 2^3 - 2 = 16 - 24 - 2 = -10$

d $P(-2) = (-2)^4 - 3 \times (-2)^3 - 2 = 16 + 24 - 2 = 38$

- A polynomial function is a function that can be written in the form:
P(x) = ax^n + ax^n-1 + ... + ax + a

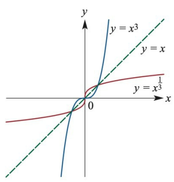
- The leading term, ax^n, of a polynomial is the term of the highest index among those terms with a non-zero coefficient.

The function f(x)=x^1/3

The function $f: \mathbb{R} \to \mathbb{R}, f(x) = x^{\frac{1}{3}}$

The functions with rules of the form $f(x) = a(x-h)^3 + k$ are one-to-one functions. Hence each of these functions has an inverse function.

The inverse function of $f(x) = x^3$ is $f^{-1}(x) = x^{\frac{1}{3}}$.



The graphs of $y = x^3$ and $y = x^{\frac{1}{3}}$ are shown above.

The graph of $y = x^{\frac{1}{3}}$ is instantaneously vertical at $x = 0$. The graphs of $y = x^3$ and $y = x^{\frac{1}{3}}$ intersect at $(1, 1)$ and $(-1, -1)$.

Example 22

Sketch the graph of $y = (x-1)^{\frac{1}{3}} - 2$.

Solution

When $x = 0$,

$y = (-1)^{\frac{1}{3}} - 2 = -1 - 2 = -3$

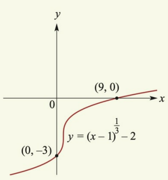
When $y = 0$,

$(x-1)^{\frac{1}{3}} - 2 = 0$

$(x-1)^{\frac{1}{3}} = 2$

$x - 1 = 2^3$

$x = 9$



Explanation

Find the axis intercepts.

The graph of $y = (x-1)^{\frac{1}{3}} - 2$ is the graph of $y = x^{\frac{1}{3}}$ translated 1 unit to the right and 2 units down.

Inverse functions

f(x)=x^1/3 is the inverse of f(x)= x 3

6E - Cubic Functions of f(x) = a(x-h)^3 + k

Example 21

Sketch the graph of the function $y = (x-2)^3 + 4$.

Solution

The graph of $y = x^3$ is translated 2 units to the right and 4 units up.

Point of inflection is $(2, 4)$.

x-axis intercept:

Let $y = 0$

$0 = (x-2)^3 + 4$

$-4 = (x-2)^3$

$\sqrt[3]{-4} = x - 2$

$x = 2 + \sqrt[3]{-4} \approx 0.413$

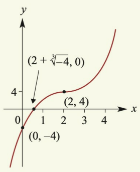
y-axis intercept:

Let $x = 0$

$y = (0-2)^3 + 4$

$y = -8 + 4$

$y = -4$



6E - Cubic Functions of f(x) = a(x-h)^3 + k

General form

For the graph of a cubic function of the form

$$y = a(x-h)^3 + k$$

the point of inflection is at (h, k).

6E - Cubic Functions of f(x) = a(x-h)^3 + k

Point of Inflection (POI)	Vertical Translations	Horizontal Translations
A point of zero gradient	by adding or subtracting a constant term to y-x^3, the graph moves either up or down	The graph of y=(x-h)^3 is moved h units to the right (for h>0)
The 'flat point' of the graph	e.g. y=x^3 + k moves the graph k units up (for k>0)	the POI is at (h,0)

6E - Cubic Functions of f(x) = a(x-h)^3 + k (cont)

The POI of graph of y=x^3 is at (0,0)	The POI becomes (0,k)	In this case, the graph of y=x^3 is translated h units in the positive direction of the x-axis.

The implied domain of all cubics is R and the range is also R

6D - Solving Cubic Equations

Example 17

Solve each of the following equations for x:

a $2x^3 - x^2 - x = 0$ **b** $x^3 + 2x^2 - 10x = 0$

Solution

a

$2x^3 - x^2 - x = 0$

$x(2x^2 - x - 1) = 0$

$x(2x+1)(x-1) = 0$

$\therefore\ x = 0$ or $x = -\frac{1}{2}$ or $x = 1$

b

$x^3 + 2x^2 - 10x = 0$

$x(x^2 + 2x - 10) = 0$

$x(x^2 + 2x + 1 - 11) = 0$

$x(x + 1 - \sqrt{11})(x + 1 + \sqrt{11}) = 0$

$\therefore\ x = 0$ or $x = -1 + \sqrt{11}$ or $x = -1 - \sqrt{11}$

6D - Solving Cubic Equations

Example 19

Solve $x^3 - 4x^2 - 11x + 30 = 0$.

Solution

Let $P(x) = x^3 - 4x^2 - 11x + 30$

Then $P(1) = 1 - 4 - 11 + 30 \neq 0$

$P(-1) = -1 - 4 + 11 + 30 \neq 0$

$P(2) = 8 - 16 - 22 + 30 = 0$

$\therefore x - 2$ is a factor.

By division or inspection,

$x^3 - 4x^2 - 11x + 30 = (x-2)(x^2 - 2x - 15) = (x-2)(x-5)(x+3)$

$\therefore (x-2)(x-5)(x+3) = 0$

$\therefore x - 2 = 0$ or $x - 5 = 0$ or $x + 3 = 0$

$\therefore x = 2, 5$ or -3

Explanation

In this example we first identify a linear factor using the factor theorem.

The factorisation is completed using one of the methods given in the previous section.

In order to solve a cubic equation, the first step is often to factorise.
Factorise by identifying a factor, then using polynomial division.
Factor the quadratic factor
Then, use null factor law.

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6B - Division of Polynomials

Example 5

Divide $x^3 + x^2 - 14x - 24$ by $x + 2$.

Solution

$$\begin{array}{r} x^2 - x - 12 \\ x+2 \overline{) x^3 + x^2 - 14x - 24} \\ x^3 + 2x^2 \\ \hline -x^2 - 14x - 24 \\ -x^2 - 2x \\ \hline -12x - 24 \\ -12x - 24 \\ \hline 0 \end{array}$$

Explanation

- Divide x, from $x + 2$, into the leading term x^3 to get x^2.
- Multiply x^2 by $x + 2$ to give $x^3 + 2x^2$.
- Subtract from $x^3 + x^2 - 14x - 24$, leaving $-x^2 - 14x - 24$.
- Now divide x, from $x + 2$, into $-x^2$ to get $-x$.
- Multiply $-x$ by $x + 2$ to give $-x^2 - 2x$.
- Subtract from $-x^2 - 14x - 24$, leaving $-12x - 24$.
- Divide x into $-12x$ to get -12.
- Multiply -12 by $x + 2$ to give $-12x - 24$.
- Subtract from $-12x - 24$, leaving remainder of 0.

In this example we see that $x + 2$ is a factor of $x^3 + x^2 - 14x - 24$, as the remainder is zero. Thus $(x^3 + x^2 - 14x - 24) \div (x + 2) = x^2 - x - 12$ with zero remainder.

$$\therefore \quad \frac{x^3 + x^2 - 14x - 24}{x + 2} = x^2 - x - 12$$

- When we divide the polynomial P(x) by the polynomial D(x) we obtain two polynomials, Q(x) the quotient and R(x) the remainder, such that

P(x) = D(x)Q(x) + R(x)

and either R(x) =0 or R9x) has degree less than D(x)

Here P(x) is the dividend and D(x) is the divisor

6B - Division of Polynomials

b $x^3 - 3x^2 + 1, \ 3x - 1$

$$\begin{array}{r} \frac{1}{3}x^2 - \frac{8}{9}x - \frac{8}{27} + \frac{19}{27(3x-1)} \\ \textbf{b} \quad 3x - 1 \overline{) x^3 - 3x^2 + 0x + 1} \\ x^3 - \frac{1}{3}x^2 \\ \hline -\frac{8}{3}x^2 + 0x \\ -\frac{8}{3}x^2 + \frac{8}{9}x \\ \hline -\frac{8}{9}x + 1 \\ -\frac{8}{9}x + \frac{8}{27} \\ \hline \frac{19}{27} \end{array}$$

Dividing Polynomials involving fractions

6B - Division of Polynomials

Example 6

Divide $3x^3 + x - 3$ by $x - 2$.

Solution

$$\begin{array}{r} 3x^2 + 6x + 13 \\ x-2 \overline{) 3x^3 + 0x^2 + x - 3} \\ 3x^3 - 6x^2 \\ \hline 6x^2 + x - 3 \\ 6x^2 - 12x \\ \hline 13x - 3 \\ 13x - 26 \\ \hline 23 \end{array}$$

Explanation

Here there is no term in x^2, however we can rewrite the polynomial as $3x^3 + 0x^2 + x - 3$.

- Divide x, from $x - 2$, into $3x^3$ to get $3x^2$.
- Multiply $3x^2$ by $x - 2$ to give $3x^3 - 6x^2$.
- Subtract from $3x^3 + 0x^2 + x - 3$, leaving $6x^2 + x - 3$.
- Now divide x, from $x - 2$, into $6x^2$ to get $6x$.
- Multiply $6x$ by $x - 2$ to give $6x^2 - 12x$.
- Subtract from $6x^2 + x - 3$, leaving $13x - 3$.
- Divide x into $13x$ to get 13.
- Multiply 13 by $x - 2$ to give $13x - 26$.
- Subtract from $13x - 3$, leaving remainder of 23.

From this example, we have

$$3x^3 + x - 3 = (x - 2)(3x^2 + 6x + 13) + 23$$

Alternatively, we can write

$$\frac{3x^3 + x - 3}{x - 2} = 3x^2 + 6x + 13 + \frac{23}{x - 2}$$

Dividing polynomials when we have a remainder

6B - Division of Polynomials

Equating coefficients to divide

We will briefly outline how to carry out divisions by equating coefficients as shown in the first section of this chapter.

To divide $x^3 - 7x^2 + 5x - 4$ by $x - 3$, first write the identity

$$x^3 - 7x^2 + 5x - 4 = (x - 3)(x^2 + bx + c) + r$$

We first find b, then c and finally r by equating coefficients of the left-hand side and right-hand side of this identity.

x^2 term Left-hand side: $-7x^2$. Right-hand side: $-3x^2 + bx^2 = (-3 + b)x^2$.
Therefore $-3 + b = -7$. Hence $b = -4$.

x term Left-hand side: $5x$. Right-hand side: $12x + cx = (12 + c)x$.
Therefore $12 + c = 5$. Hence $c = -7$.

constant term Left-hand side: -4. Right-hand side: $21 + r$.
Therefore $21 + r = -4$. Hence $r = -25$.

So we can write

$$x^3 - 7x^2 + 5x - 4 = (x - 3)(x^2 - 4x - 7) - 25$$

Equating Coefficients Methods instead of dividing

6F - Graphs of factorised cubic functions

Example 19

Solve $x^3 - 4x^2 - 11x + 30 = 0$.

Solution

Let $P(x) = x^3 - 4x^2 - 11x + 30$

Then $P(1) = 1 - 4 - 11 + 30 \neq 0$

$P(-1) = -1 - 4 + 11 + 30 \neq 0$

$P(2) = 8 - 16 - 22 + 30 = 0$

$\therefore x - 2$ is a factor.

By division or inspection,

$$x^3 - 4x^2 - 11x + 30 = (x - 2)(x^2 - 2x - 15) = (x - 2)(x - 5)(x + 3)$$

$\therefore (x - 2)(x - 5)(x + 3) = 0$

$\therefore x - 2 = 0$ or $x - 5 = 0$ or $x + 3 = 0$

$\therefore x = 2, 5$ or -3

Explanation

In this example we first identify a linear factor using the factor theorem.

The factorisation is completed using one of the methods given in the previous section.

6F - Graphs of factorised cubic functions

Repeated factors

The polynomial function $f(x) = (x - 1)^2(x + 3)$ has a **repeated factor**. In this case $x - 1$ is repeated. Since the repeated factor is squared, it is easy to see that the sign of the y-value is the same 'close in' on either side of the corresponding x-axis intercept.

If the factorised cubic has a repeated factor and another linear factor, there are only two x-axis intercepts and the repeated factor corresponds to one of the turning points.

Example 26

Sketch the graph of $y = x^2(x - 1)$.

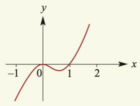
Solution

To find the x-axis intercepts, let $y = 0$.

Then $x^2(x - 1) = 0$.

Thus the x-axis intercepts are at $x = 0$ and $x = 1$.
Because the repeated factor is x^2, there is also a turning point at $x = 0$.

The y-axis intercept (letting $x = 0$) is at $y = 0$.



Repeated roots/factors

6F - Graphs of factorised cubic functions

Cubics with one x-axis intercept

Cubics of the form $y = (x - a)^3$ have only one x-axis intercept. Some other cubics also have only one x-axis intercept because, when they are factorised, they are found to have only one linear factor, with the remaining quadratic factor unable to be factorised further.

Example 27

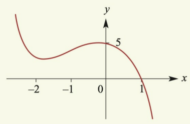
Sketch the graph of $y = -(x - 1)(x^2 + 4x + 5)$.

Solution

To find the x-axis intercept, let $y = 0$.

First, we note that the factor $x^2 + 4x + 5$ cannot be factorised further:

$$\Delta = b^2 - 4ac = 4^2 - 4(1)(5) = -4$$



$\therefore$ there are no further linear factors.

Hence, when solving the equation $-(x - 1)(x^2 + 4x + 5) = 0$, there is only one solution.

$\therefore x$-axis intercept is $x = 1$.

To find the y-axis intercept, let $x = 0$. Then $y = -(0 - 1)(0^2 + 4(0) + 5) = 5$.

A CAS calculator can be used to find the turning points $(0, 5)$ and $(-1.82, 2.91)$, where the coordinates of the second point are given to two decimal places.

Cubic equations with one x intercept

6G - Solving Cubic Inequalities

Example 28

Find $\{ x : x^3 + x^2 - 5x + 3 \leq 0 \}$.

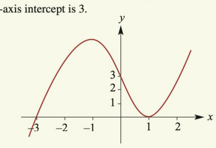
Solution

$P(x) = x^3 + x^2 - 5x + 3$

$P(1) = 1 + 1 - 5 + 3 = 0$

$\therefore x - 1$ is a factor. By division, $y = (x - 1)^2(x + 3)$.

There are only two x-axis intercepts, 1 and -3. The y-axis intercept is 3.



From the graph we can see that $y \leq 0$ when $x \leq -3$ or when $x = 1$.

$$\therefore \quad \{ x : x^3 + x^2 - 5x + 3 \leq 0 \} = (-\infty, -3] \cup \{1\}$$

Explanation

- Use the factor theorem to find an initial linear factor.
- Complete the factorisation.
- Find the axis intercepts.
- Sketch the graph. (Note that this cubic has a repeated factor, and so $(1, 0)$ is a turning point as well an intercept.)
- Solve the inequality by inspecting the graph.
- Express the solution in formal set notation.

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6H - Families of cubic polynomial functions

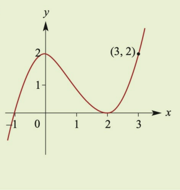
y=ax^3	y=a(x-h)^3 + k	y=a(x-a)(x-b)(x-c)	y=ax^3-+bx-^2+cx+d

6H - Families of cubic polynomial functions

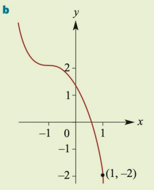
Example 31

Determine the rule for the cubic function shown in each of the following graphs:

a



b



Solution

a $y = a(x+1)(x-2)^2$

Put (3, 2) into the equation:

$2 = a(4)(1)$

$\frac{1}{2} = a$

The rule is $y = \frac{1}{2}(x+1)(x-2)^2$.

b $y = a(x+1)^3 + 2$

To determine a, put the known point (1, −2) into the equation:

$-2 = a(2)^3 + 2$

$-4 = 8a$

$-\frac{1}{2} = a$

The rule is $y = -\frac{1}{2}(x+1)^3 + 2$.

Explanation

The x-axis intercepts are −1 and 2, and the graph touches the x-axis at 2. So the cubic has a repeated factor $x - 2$.

Therefore the form of the rule appears to be $y = a(x+1)(x-2)^2$.

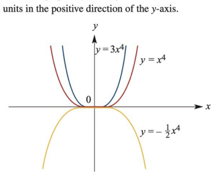
This graph appears to be of the form $y = a(x-h)^3 + k$. The point of inflection is at (−1, 2). Therefore $h = -1$ and $k = 2$.

6H - Families of cubic polynomial functions

Quartic functions of the form $f(x) = a(x-h)^4 + k$

The graph of $f(x) = (x-1)^4 + 3$ is obtained from the graph of $y = x^4$ by a translation of 1 unit in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

As with other graphs it has been seen that changing the value of a simply narrows or broadens the graph without changing its fundamental shape. Again, if $a < 0$, the graph is inverted.



The significant feature of the graph of a quartic of this form is the **turning point** (a point of zero gradient). The turning point of $y = x^4$ is at the origin (0, 0).

For the graph of a quartic function of the form

$$y = a(x-h)^4 + k$$

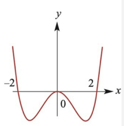
the turning point is at (h, k).

When sketching quartic graphs of the form $y = a(x-h)^4 + k$, first identify the turning point. To add further detail to the graph, find the x-axis and y-axis intercepts.

Other quartic functions

The techniques for graphing quartic functions in general are very similar to those employed for cubic functions. A CAS calculator is to be used in the graphing of these functions. Great care needs to be taken in this process as it is easy to miss key points on the graph using these techniques.

The graph of $y = 2x^4 - 8x^2$ is shown.



Quartic Functions

6B - Division of Polynomials

Equating coefficients to divide

We will briefly outline how to carry out divisions by equating coefficients as shown in the first section of this chapter.

To divide $x^3 - 7x^2 + 5x - 4$ by $x - 3$, first write the identity

$$x^3 - 7x^2 + 5x - 4 = (x-3)(x^2 + bx + c) + r$$

We first find b, then c and finally r by equating coefficients of the left-hand side and right-hand side of this identity.

x^2 term Left-hand side: $-7x^2$. Right-hand side: $-3x^2 + bx^2 = (-3+b)x^2$.
Therefore $-3 + b = -7$. Hence $b = -4$.

x term Left-hand side: $5x$. Right-hand side: $12x + cx = (12+c)x$.
Therefore $12 + c = 5$. Hence $c = -7$.

constant term Left-hand side: −4. Right-hand side: $21 + r$.
Therefore $21 + r = -4$. Hence $r = -25$.

So we can write

$$x^3 - 7x^2 + 5x - 4 = (x-3)(x^2 - 4x - 7) - 25$$

Equating Coefficients Methods instead of dividing

6C - Special Cases Differences of Cubes

Example 14

Factorise $x^3 - 27$.

Solution

Let $P(x) = x^3 - 27$

Then $P(3) = 27 - 27 = 0$

Thus $x - 3$ is a factor.

Divide to find the other factor:

$$\begin{array}{r} x^2 + 3x + 9 \\ x - 3 \,\big)\, x^3 + 0x^2 + 0x - 27 \\ x^3 - 3x^2 \\ \hline 3x^2 + 0x - 27 \\ 3x^2 - 9x \\ \hline 9x - 27 \\ 9x - 27 \\ \hline 0 \end{array}$$

Hence

$$x^3 - 27 = (x-3)(x^2 + 3x + 9)$$

Alternative

The division can also be performed using the method of equating coefficients.

Let $x^3 - 27 = (x-3)(x^2 + bx + c)$.

Equating constant terms gives $c = 9$.

Equating coefficients of x^2 gives $-3 + b = 0$, and so $b = 3$.

Hence $x^3 - 27 = (x-3)(x^2 + 3x + 9)$.

6C - Rational Root Theorem

Example 13

Factorise $P(x) = 3x^3 + 8x^2 + 2x - 5$.

Solution

$P(1) = 8 \neq 0$, $P(-1) = -2 \neq 0$,

$P(5) = 580 \neq 0$, $P(-5) = -190 \neq 0$,

$P\left(-\frac{5}{3}\right) = 0$

Therefore $3x + 5$ is a factor.

Dividing gives

$$P(x) = 3x^3 + 8x^2 + 2x - 5 = (3x+5)(x^2 + x - 1)$$

We complete the square for $x^2 + x - 1$ to factorise:

$$x^2 + x - 1 = x^2 + x + \frac{1}{4} - \frac{1}{4} - 1 = \left(x + \frac{1}{2}\right)^2 - \frac{5}{4} = \left(x + \frac{1}{2} + \frac{\sqrt{5}}{2}\right)\left(x + \frac{1}{2} - \frac{\sqrt{5}}{2}\right)$$

Hence

$$P(x) = (3x+5)\left(x + \frac{1}{2} + \frac{\sqrt{5}}{2}\right)\left(x + \frac{1}{2} - \frac{\sqrt{5}}{2}\right)$$

Explanation

The only possible integer solutions are ±5 or ±1. So there are no integer solutions. We now use the rational-root theorem.

If $-\frac{\alpha}{\beta}$ is a solution, the only value of β that needs to be considered is 3 and $\alpha = \pm 5$ or $\alpha = \pm 1$.

6C - Rational Root Theorem

P(x) = $2x^{3 - x}2$ - x - 3

Choose factors of -3, which are ±1 and ±3.

However, P(1) ≠ 0, P(-1) ≠ 0, P(3) ≠ 0, and P(-3) ≠ 0

Therefore, we must use the Rational Root Theorem.

We must use P(± factors of constant/factors of leading coefficient)

factors of constant (of -3) = ±1, ±3 factors leading coefficient (of 2) = ±1, ±2

e.g. P(±3/2), P(±1/2)

We now have to check these factors --> P(3/2) = $2(3/2)^{3 - (3/2)}2$ -(3/2) -3 = 0

therefore, x-3/2, which equates to 2x-3 as a factor

6C - Factorisation of Polynomials

Example 12

Factorise $x^3 - 2x^2 - 5x + 6$.

Solution

$P(1) = 1 - 2 - 5 + 6 = 0$

$\therefore x - 1$ is a factor.

Now divide to find the other factors:

$$\begin{array}{r} x^2 - x - 6 \\ x - 1 \,\big)\, x^3 - 2x^2 - 5x + 6 \\ x^3 - x^2 \\ \hline -x^2 - 5x + 6 \\ -x^2 + x \\ \hline -6x + 6 \\ -6x + 6 \\ \hline 0 \end{array}$$

$$\therefore x^3 - 2x^2 - 5x + 6 = (x-1)(x^2 - x - 6) = (x-1)(x-3)(x+2)$$

Explanation

The factors of 6 are ±1, ±2, ±3, ±6.

We evaluate the first option, $P(1)$, which in fact equals 0. If $P(1)$ did not equal 0, we would try the other factors of 6 in turn until a zero result is found.

For a polynomial P(x)
- If P(a) = 0, then x-a is a factor of P(x)
- Conversely, if x-a is a factor of P(x), then P(a) =0

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6C - Factorisation of Polynomials

Example 9

Use the remainder theorem to find the value of the remainder when:

a $P(x) = x^3 - 3x^2 + 2x + 6$ is divided by $x - 2$

b $P(x) = x^3 - 2x + 4$ is divided by $2x + 1$.

Solution	Explanation
a $P(2) = (2)^3 - 3(2)^2 + 2(2) + 6$ $= 8 - 12 + 4 + 6$ $= 6$ The remainder is 6.	We apply the remainder theorem by evaluating $P(2)$.
b $P\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^3 - 2\left(-\frac{1}{2}\right) + 4$ $= -\frac{1}{8} + 1 + 4$ $= \frac{39}{8}$ The remainder is $\frac{39}{8}$.	We apply the remainder theorem by evaluating $P(-\frac{1}{2})$.

Example 10

When $P(x) = x^3 + 2x + a$ is divided by $x - 2$, the remainder is 4. Find the value of a.

Solution	Explanation
$P(2) = 8 + 4 + a = 4$ Therefore $a = -8$.	We apply the remainder theorem to form a linear equation in a.

Remainder Theorem:
When P(x) is divided by bx+a, the remainder is P(-a/b).

For example, if P(x) is divided by x-1, let x-1=0, x=1.
P(1) = Remainder (R(x))

For example, if P(x) is divided by 3x-2, let 3x+2=0, x=-2/3.
P(-2/3) = Remainder (R(x))

6J - Applications

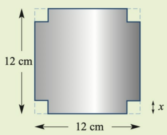
Example 37

A square sheet of tin measures 12 cm × 12 cm.

Four equal squares of edge x cm are cut out of the corners and the sides are turned up to form an open rectangular box. Find:

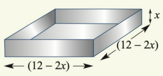
a the values of x for which the volume is 100 cm^3

b the maximum volume.



Solution

The figure shows how it is possible to form many open rectangular boxes with dimensions $12 - 2x$, $12 - 2x$ and x.



The volume of the box is

$$V = x(12 - 2x)^2, \quad 0 \le x \le 6$$

which is a cubic model.

We complete the solution using a CAS calculator as follows.

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