4A - Rectangular Hyperbolas

Brief Summary

$$y = \frac{a}{x-h} + k$$

Asymptotes are the lines $y = k$ and $x = h$

This is the standard form of a rectangular hyperbola:
a dilates the graph
if the graph is negative, it is reflected along the x axis.
h moves the graph left and right (along x axis)
k moves the graph up and down (along y axis)
To find asymptotes,
y=k, horizontal asymptote
x=h vertical asymptote (or set x-h=0, and solve for x).

4A - Sketching Rectangular Hyperbolas

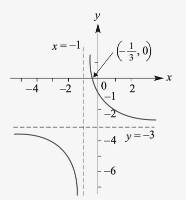
Sketching rectangular hyperbolas

Using dilations, reflections and translations, we are now able to sketch the graphs of all rectangular hyperbolas of the form $y = \frac{a}{x-h} + k$.

Example 1

Sketch the graph of $y = \frac{2}{x+1} - 3$.

Solution



Explanation

The graph of $y = \frac{2}{x}$ has been translated 1 unit to the left and 3 units down. The asymptotes have equations $x = -1$ and $y = -3$.

When $x = 0$, $y = \frac{2}{0+1} - 3 = -1$.

$\therefore$ the y-axis intercept is -1.

When $y = 0$,

$$0 = \frac{2}{x+1} - 3$$
$$3 = \frac{2}{x+1}$$
$$3(x+1) = 2$$
$$x = -\frac{1}{3}$$

$\therefore$ the x-axis intercept is $-\frac{1}{3}$.

First sketch the graph along asymptotes to visual whether there is a x and/or y intercept.
Then solve for x and/or y intercept and label.

4A - Rearranging Form of Rectangular Hyperbolas.

First, place the denominators also as the numerator.
Then, use an appropriate coefficient outside brackets to expand to the correct x value.
Once expanded, select an appropriate number to equate to the constant.
Seperate the constant over its own fraction, and cancel out the numerator and denominator.

7D - Determining Transformations

Summary 7D

The notation developed in this chapter can be used to help find the transformation that takes the graph of a function to its image.

For example, if the graph of $y = f(x)$ is mapped to the graph of $y' = 2f(x' - 3)$, we can see that the transformation

$$x' = x + 3 \quad \text{and} \quad y' = 2y$$

is a suitable choice. This is a translation of 3 units to the right followed by a dilation of factor 2 from the x-axis.

There are infinitely many transformations that take the graph of $y = f(x)$ to the graph of $y' = 2f(x' - 3)$. The one we chose is conventional.

e. Find a sequence of transformations that takes the graph of $y = 2\sqrt{4-x} + 3$ to the graph of $y = -\sqrt{x} + 6$.

$y = 2\sqrt{4-x} + 3 \rightarrow y' = -\sqrt{x'} + 6$

$\frac{y-3}{2} = \sqrt{4-x}$

$-y' + 6 = \sqrt{x'}$

$-(x - 4)$

$\frac{y-3}{2} = -y' + 6$

$4 - x = x'$

$-\left(\frac{y-3}{2} - 6\right) = y'$

$-x + 4 = x'$

reflect in the y axis
4 right

$-\left(\frac{y-3}{2} - \frac{12}{2}\right) = y'$

$-\left(\frac{y-15}{2}\right) = y'$

15 down
$\frac{1}{2}$ from x
reflect

$-\frac{y}{2} + \frac{15}{2}$

$\frac{1}{2}$ from x
reflect
$\frac{15}{2}$ up

7C - Function Notation and Transformations

Mapping	Rule	The graph of $y = f(x)$ maps to
Reflection in the x-axis	$(x, y) \rightarrow (x, -y)$	$y = -f(x)$
Reflection in the y-axis	$(x, y) \rightarrow (-x, y)$	$y = f(-x)$
Dilation of factor a from the y-axis	$(x, y) \rightarrow (ax, y)$	$y = f\left(\frac{x}{a}\right)$
Dilation of factor b from the x-axis	$(x, y) \rightarrow (x, by)$	$y = bf(x)$
Translation of h units in the positive direction of the x-axis and k units in the positive direction of the y-axis	$(x, y) \rightarrow (x + h, y + k)$	$y - k = f(x - h)$
Reflection in the line $y = x$	$(x, y) \rightarrow (y, x)$	$x = f(y)$

7C- Combinations of Transformations

Brief Summary

To **APPLY** a transformation, solve for x and y then sub in,

A sequence of transformations in the order given:
1 – a dilation of factor 2 from the x-axis
2 – a reflection in the x-axis
May be described by the rule:

$$(x, y) \rightarrow (x, 2y) \rightarrow (x, -2y)$$

(Note, here the order of the transformations didn't matter – but often they do).

A sequence of transformations in the order given:
1 – a dilation of factor 2 from the x-axis
2 – a translation of 3 units in the positive direction of the y-axis
May be described by the rule:

$$(x, y) \rightarrow (x, 2y) \rightarrow (x, 2y + 3)$$

A sequence of transformations in the order given:
1 – a translation of 3 units in the positive direction of the y-axis
2 – a dilation of factor 2 from the x-axis
May be described by the rule:

$$(x, y) \rightarrow (x, y + 3) \rightarrow (x, 2(y + 3))$$

Summary 7C

Given a sequence of transformations, we can find the rule for transforming points of the plane. For example, the sequence

- a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis
- followed by a reflection in the y-axis

can be described by the rule $(x, y) \rightarrow (x + 2, y - 3) \rightarrow (-x - 2, y - 3)$.

NOTE: the order in which the transformations are applied are very important
as above, use brackets to follow order of operations.

7B - Dilations and Reflections

Brief Summary

For a positive constant b, a dilation of b units from the x-axis is described by either of the following:
$(x, y) \rightarrow (x, by)$
$x' = x, \quad y' = by$
Applying the transformation $(x, y) \rightarrow (x, by)$ to $y = f(x)$ may be completed by replacing x with x and y with $\frac{y}{b}$, obtaining $\frac{y}{b} = f(x)$

For a positive constant a, a dilation of a units from the y-axis is described by either of the following:
$(x, y) \rightarrow (ax, y)$
$x' = ax, \quad y' = y$
Applying the transformation $(x, y) \rightarrow (ax, y)$ to $y = f(x)$ may be completed by replacing x with $\frac{x}{a}$ and y with y, obtaining $y = f(\frac{x}{a})$

A reflection in the x-axis is described by the rule $(x, y) \rightarrow (x, -y)$
A reflection in the y-axis is described by the rule $(x, y) \rightarrow (-x, y)$

1) Determine the rule for the image when the graph of $y = \frac{1}{x^2}$ is dilated by a factor 4:

a) From the y-axis

$(x, y) \rightarrow (4x, y)$

$x' = 4x \quad y' = y$

$x = \frac{1}{4}x' \quad y = y'$

$y = \frac{1}{x^2}$ becomes:

$y' = \frac{1}{(\frac{1}{4}x')^2}$

$y = \frac{1}{(\frac{1}{16}x^2)} = \frac{16}{x^2}$

b) From the x-axis

$(x, y) \rightarrow (x, 4y)$

$x' = x \quad y' = 4y$

$x = x' \quad y = \frac{1}{4}y'$

$y = \frac{1}{x^2}$ becomes:

$\frac{1}{4}y' = \frac{1}{(x')^2}$

$y = \frac{4}{x^2}$

7A - Translations

Brief Summary

A translation of h units in the positive direction of the x-axis and k units in the positive direction of the y-axis is described by either of the following:
$(x, y) \to (x + h, y + k)$
$x' = x + h, \qquad y' = y + k$
Applying the transformation $(x, y) \to (x + h, y + k)$ to $y = f(x)$ may be completed by replacing x with $x - h$ and y with $y - k$, obtaining $y - k = f(x - h)$

A translation of h units in the positive direction of the x-axis and k units in the positive direction of the y-axis is described by the rule

$$(x, y) \to (x + h, y + k)$$

or $\quad x' = x + h \quad$ and $\quad y' = y + k$

where h and k are positive numbers.

A translation of h units in the negative direction of the x-axis and k units in the negative direction of the y-axis is described by the rule

$$(x, y) \to (x - h, y - k)$$

or $\quad x' = x - h \quad$ and $\quad y' = y - k$

where h and k are positive numbers.

Example 1

Find the equation for the image of the curve with equation $y = f(x)$, where $f(x) = \frac{1}{x}$, under a translation 3 units in the positive direction of the x-axis and 2 units in the negative direction of the y-axis.

Solution	Explanation
Let (x', y') be the image of the point (x, y), where (x, y) is a point on the graph of $y = f(x)$.	The rule is $(x, y) \to (x + 3, y - 2)$.
Then $x' = x + 3$ and $y' = y - 2$.	
Hence $x = x' - 3$ and $y = y' + 2$.	
The graph of $y = f(x)$ is mapped to the graph of $y' + 2 = f(x' - 3)$	Substitute $x = x' - 3$ and $y = y' + 2$ into $y = f(x)$.
i.e. $y = \frac{1}{x}$ is mapped to $y' + 2 = \frac{1}{x' - 3}$	

4B - The Truncus

Brief Summary

$$y = \frac{a}{(x-h)^2} + k$$

Asymptotes are the lines $y = k$ and $x = h$

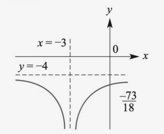
This is the standard form of a truncus:
a dilates the graph
if the graph is negative, it is reflected along the x axis.
h moves the graph left and right (along x axis)
k moves the graph up and down (along y axis)
To find asymptotes,
y=k, horizontal asymptote
x=h vertical asymptote (or set x-h=0, and solve for x).

4B - Sketching a Truncus

Example 3

Sketch the graph of $y = \frac{-1}{2(x+3)^2} - 4$.

Solution



Explanation

The graph of $y = \frac{-1}{2x^2}$ is translated 3 units to the left and 4 units down.

The y-axis intercept is found by putting $x = 0$; it is $\frac{-73}{18}$.

First sketch the graph along asymptotes to visual whether there is a x and/or y intercept.
Then solve for x and/or y intercept and label.

4D - The Graph of y= √x

Brief Summary

$$y = a\sqrt{x-h} + k$$

An endpoint will occur at (h, k)

Summary 4D

- All graphs of the form $y = a\sqrt{x-h} + k$ will have the same basic shape as the graph of $y = \sqrt{x}$. The graph will have endpoint (h, k).
- The graph of $y = \sqrt{-x}$ is the reflection in the y-axis of the graph of $y = \sqrt{x}$.

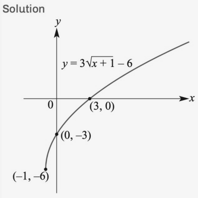
The general form of a root x graph:
a dilates the graph
if a is negative (-a), the graph is reflected in the x axis
if x is negative (√-x), the graph is reflected in the x axis
NOTE: ensure to rearrange if there is a -x
h moves the graph left and right (along the x axis)
k moves the graph up and down (along the y axis)
An endpoint will occur at (h,k) - reverse symbol of h value.

4D - Sketching a graph of y=√x

Example 5

Sketch the graph of $y = 3\sqrt{x+1} - 6$.

Solution



When $x = 0$, $y = -3$

When $y = 0$, $3\sqrt{x+1} - 6 = 0$
$3\sqrt{x+1} = 6$
$\sqrt{x+1} = 2$
$x + 1 = 4$
$x = 3$

Explanation

The graph is formed by dilating the graph of $y = \sqrt{x}$ from the x-axis by factor 3 and then translating 1 unit to the left and 6 units down.

The rule is defined for $x \geq -1$.

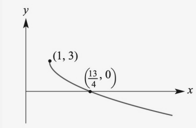
The set of values the rule can take (the range) is all numbers greater than or equal to -6, i.e. $y \geq -6$.

4D - Sketching a graph of y=-√x

Example 6

Sketch the graph of $y = -2\sqrt{x-1} + 3$.

Solution



When $y = 0$: $\quad -2\sqrt{x-1} + 3 = 0$
$2\sqrt{x-1} = 3$
Square both sides: $\quad 4(x-1) = 9$
Therefore $\quad x = \frac{9}{4} + 1 = \frac{13}{4}$

Explanation

The graph is formed by dilating the graph of $y = \sqrt{x}$ from the x-axis by factor 2, reflecting this in the x-axis and then translating it 1 unit to the right and 3 units up.

The rule is defined for $x \geq 1$.

The set of values the rule can take (the range) is all numbers less than or equal to 3, i.e. $y \leq 3$.

A y=-√x graph is reflected in the X AXIS.

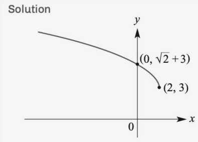
4D - Sketching a y=√-x graph

Example 7

Sketch the graph of $y = \sqrt{2-x} + 3$.

Note: $\sqrt{2-x} = \sqrt{-(x-2)}$

Solution



When $x = 0$, $y = \sqrt{2} + 3$.

Explanation

We can write the rule as

$$y = \sqrt{-(x-2)} + 3$$

The rule is defined for $x \leq 2$. The set of values the rule can take (the range) is all numbers greater than or equal to 3, i.e. $y \geq 3$.

A y=√-x graph is reflected in the Y AXIS.
When sketching a y=√-x graph, sometimes it will appear as a y=√h-x graph, ensure it is rearranged so that the -1 coefficient is outside the brackets -> as y=√-(x-h) graph

5H - Applications

Example 26

A householder has six laying hens and wishes to construct a rectangular enclosure to provide a maximum area for the hens, using a 12 m length of fencing wire. Construct a function that will give the area of the enclosure, A, in terms of its length, ℓ. By sketching a graph, find the maximum area that can be fenced.

Solution

Let ℓ = length of the enclosure

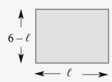
Then width $= \frac{12 - 2\ell}{2} = 6 - \ell$

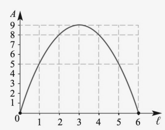
The area is

$$A(\ell) = \ell(6 - \ell)$$
$$= 6\ell - \ell^2$$

The domain of A is the interval $[0, 6]$.

The maximum area is 9 m² and occurs when $\ell = 3$ m, i.e. when the enclosure is a square.





5A - Set Notation and Sets of Numbers.

Interval notation

Among the most important subsets of $\mathbb{R}$ are the **intervals**. The following is an exhaustive list of the various types of intervals and the standard notation for them. We suppose that a and b are real numbers with $a < b$.

$(a, b) = \{x : a < x < b\}$ $\quad [a, b] = \{x : a \le x \le b\}$

$(a, b] = \{x : a < x \le b\}$ $\quad [a, b) = \{x : a \le x < b\}$

$(a, \infty) = \{x : a < x\}$ $\quad [a, \infty) = \{x : a \le x\}$

$(-\infty, b) = \{x : x < b\}$ $\quad (-\infty, b] = \{x : x \le b\}$

Note: The 'closed' circle (•) indicates that the number is included.
The 'open' circle (○) indicates that the number is not included.

The following are subsets of the real numbers for which we have special notation:

- Positive real numbers: $\mathbb{R}^+ = \{x : x > 0\}$
- Negative real numbers: $\mathbb{R}^- = \{x : x < 0\}$
- Real numbers excluding zero: $\mathbb{R} \setminus \{0\}$

Question 1

a) Write the following set using interval notation:

$\{x: -1 \le x < 7\}$

$[-1, 7)$

b) Write the following using set notation:

i) $(-\infty, 2]$

$\{x: -\infty < x \le 2\}$

ii) $\mathbb{R}^+ \cup \{0\}$ → all pos real numbers and zero

$\{x: 0 \le x < \infty\}$

5B - Function Notation

$f: x \to y, f(x) = \ldots$ → rule

domain, co-domain, function

5B - Relations, Domain and Range

Brief Summary

The domain is the set of all allowable $x-values$ (inputs)
The range is the set of all corresponding $y-values$ (outputs)

- The **maximal** or **implied domain** is the largest domain for which the rule of the relation has meaning.

Q1 Find domain and range

a. $y = \frac{4}{(2-x)^2} + 1$ ⇒ truncus graph, positive = no reflection

$y = \frac{4}{(-x+2)^2} + 1$

Find asymptotes

$x = h$, $-x+2 = 0$ ⇒ $x = 2$

$\frac{-x}{-1} = \frac{-2}{-1}$ $x = 2$

$y = k$, $y = 1$

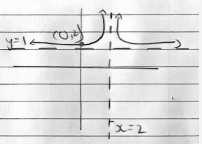
Find y inter, let $x = 0$

$y = \frac{4}{(-0+2)^2} + 1$

$y = \frac{4}{(2)^2} + 1 \Rightarrow \frac{4}{4} + 1 = 1 + 1 = 2$ $(0, 2)$

∴ Domain $= \mathbb{R} \setminus \{2\}$, all real numbers excluding

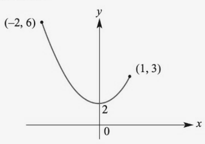
Range $= (1, \infty)$, 1 to positive infinity.



Example 6

Sketch the graph of the relation $y = x^2 + 2$ for $x \in [-2, 1]$ and state the range.

Solution



The range is $[2, 6]$.

Explanation

Note that the range is not determined by considering the endpoints alone. The minimum y-value is 2, not 3.

5C - Functions

Vertical-line test

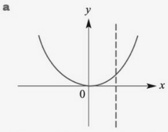
If a vertical line can be drawn anywhere on the graph and it only ever intersects the graph a maximum of once, then the relation is a **function**.

Example 9

a Is $y = x^2$ a function? State the maximal domain and range of $y = x^2$.
b Is $x^2 + y^2 = 4$ a function? State the maximal domain and range of $x^2 + y^2 = 4$.

Solution

a



The vertical-line test shows that $y = x^2$ is a function. The maximal domain is $\mathbb{R}$ and the range is $\mathbb{R}^+ \cup \{0\}$.

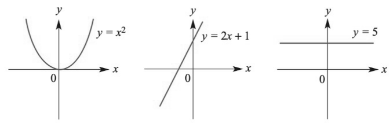
Explanation

For each x-value there is only one y-value.

The ordered pairs of the relation are all of the form (a, a^2).

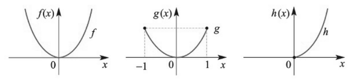
Horizontal-line test

If a horizontal line can be drawn anywhere on the graph of a function and it only ever intersects the graph a maximum of once, then the function is **one-to-one**.



Restriction of a function

Consider the following functions:



$f(x) = x^2,\ x \in \mathbb{R}$ $\quad g(x) = x^2,\ -1 \le x \le 1$ $\quad h(x) = x^2,\ x \in \mathbb{R}^+ \cup \{0\}$

The different letters, f, g and h, used to name the functions emphasise the fact that there are three different functions, even though they all have the same rule. They are different because they are defined for different domains. We say that g and h are **restrictions** of f, since their domains are subsets of the domain of f.

We can restrict a function to make it one to one.
As such, a parabola can have its domain restricted so it is a half parabola - this way it is now a one-to-one function.

5E - Piecewise Functions

Example 17

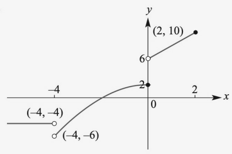
a Sketch the graph of the function:

$$f(x) = \begin{cases} -4 & \text{for } x < -4 \\ -\frac{1}{2}x^2 + 2 & \text{for } -4 < x \le 0 \\ 2x + 6 & \text{for } 0 < x \le 2 \end{cases}$$

b State the range of the function.

Solution

a



b The range is $(-6, 2] \cup (6, 10]$.

Explanation

- The graph of $y = -4$ is sketched for $x < -4$.
- The graph of $y = -\frac{1}{2}x^2 + 2$ is sketched for $-4 < x \le 0$.
- The graph of $y = 2x + 6$ is sketched for $0 < x \le 2$.

Functions which have different rules for different subsets of their domain are called piecewise-defined functions. They are also known as hybrid functions.

5F - Applying Function Notation

Example 19

If $f: \mathbb{R} \to \mathbb{R}$, $f(x) = ax + b$ such that $f(1) = 7$ and $f(5) = 19$, find a and b and sketch the graph of $y = f(x)$.

Solution

Since $f(1) = 7$ and $f(5) = 19$,

$7 = a + b$ (1)

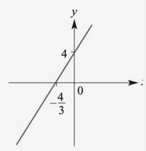
and $19 = 5a + b$ (2)

Subtract (1) from (2):

$12 = 4a$

Thus $a = 3$ and substituting in (1) gives $b = 4$.

Hence $f(x) = 3x + 4$.



Example 20

Find the quadratic function f such that $f(4) = f(-2) = 0$ and $f(0) = 16$.

Solution

Since 4 and −2 are solutions to the quadratic equation $f(x) = 0$, we have

$f(x) = k(x - 4)(x + 2)$

Since $f(0) = 16$, we obtain

$16 = k(-4)(2)$

$\therefore\ k = -2$

Hence

$f(x) = -2(x - 4)(x + 2)$

$= -2(x^2 - 2x - 8)$

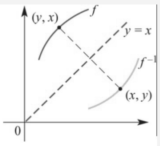
$= -2x^2 + 4x + 16$

NOTE: when evaluating piecewise functions, ensure you pay attention to the x value, and the appropriate domain for each rule.

5G - Inverse functions

Brief Summary

The inverse function $f^{-1}(x)$ may only exist if $f(x)$ is a one-to-one function.
The domain of the inverse function is the range of the original function.
The range of the original function is the domain of the inverse function.
The inverse function is given by letting $x = f(y)$ and solving for y
We may think of the inverse function as 'mapping all x-values to y-values and vice versa.
We may also think of the inverse function as the reflection of the original function through the line $y = x$



From this the following is evident:

$$\operatorname{dom} f^{-1} = \operatorname{ran} f$$
$$\operatorname{ran} f^{-1} = \operatorname{dom} f$$

A function has an inverse function if and only if it is one-to-one.

Example 23

Find the inverse of the function $f : [1, \infty) \to \mathbb{R}$, $f(x) = (x-1)^2 + 4$.

Solution

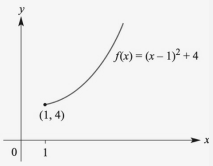
The inverse has rule

$$x = (y-1)^2 + 4$$
$$(y-1)^2 = x - 4$$
$$y - 1 = \pm\sqrt{x-4}$$
$$\therefore \quad y = 1 \pm \sqrt{x-4}$$

But $\operatorname{ran} f^{-1} = \operatorname{dom} f = [1, \infty)$, and so

$$f^{-1}(x) = 1 + \sqrt{x-4}$$

with $\operatorname{dom} f^{-1} = \operatorname{ran} f = [4, \infty)$



NOTE: when finding inverse functions always:
- Sketch the original function using its domain, and find its range.
- Sketching the original can help decide if the inverse should be positive or negative
- Once decided, then, sketch the inverse over the same set of axes, and label y=x on the graph.
- Swap the original's domain and range to then find the inverse's .

5G - Inverse Functions

8 Let $f : (-\infty, a] \to \mathbb{R}$, $f(x) = \sqrt{a-x}$, where a is a constant.
  a Find $f^{-1}(x)$.
  b If the graphs of $y = f(x)$ and $y = f^{-1}(x)$ intersect at $x = 1$, find the possible values of a.

8 $f : (-\infty, a] \to \mathbb{R}$, $f(x) = \sqrt{a-x}$

a $x = \sqrt{a-y}$

$\therefore x^2 = a - y$, $\therefore y = a - x^2$
$f^{-1}(x) = a - x^2, x \geq 0$ (to match Range of f)

b At $x = 1$: $\sqrt{a-x} = a - x^2$

$$\therefore \sqrt{a-1} = a - 1$$
$$\therefore a - 1 = (a-1)^2$$
$$\therefore a^2 - 2a + 1 - a + 1 = 0$$
$$\therefore a^2 - 3a + 2 = 0$$
$$\therefore (a-2)(a-1) = 0$$
$$\therefore a = 1, 2$$