Measure Theory

Classes of sets

Definition1.1(**σ-algebra**):A class of sets A ⊂ 2Ω if it fulfils the following three conditions: (i) Ω ∈ A. (ii) A is closed under complements. (iii) A is closed under countable unions.

Defitinition1.2(**algebra**):A class of sets A ⊂ 2Ω is called an algebra if the following three conditions are fulfilled: (i) Ω ∈ A. (ii) A is \-closed. (iii) A is ∪-closed

Definition 1.3(**ring**):A class of sets A ⊂ 2Ω is called a ring if the following three condi- tions hold: (i) ∅ ∈ A. (ii) A is \-closed. (iii) A is ∪-closed. A ring is called a **σ-ring** if it is also σ-∪-closed

Definition 1.4(**semiring**): A class of sets A ⊂ 2Ω is called a semiring if (i) ∅ ∈ A, (ii) for any two sets A, B ∈ A the difference set B \ A is a finite union of mutually disjoint sets in A, (iii) A is ∩-closed.

Definition 1.5(**Dynkin-system**):A class of sets A ⊂ 2Ω is called a λ-system (or Dynkin's λ-system) if (i) Ω ∈ A, (ii) for any two sets A, B ∈ A with A ⊂ B, the difference set B \ A is in A, and (iii) ⊎∞ n=1 An ∈ A for any choice of countably many pairwise disjoint sets A1, A2, . . . ∈ A.

Definition 1.6 (**liminf & limsup**):Let A1, A2, . . . be subsets of Ω. The sets lim inf n→∞ An := ∞∪ n=1 ∞∩ m=n Am and lim sup n→∞ An := ∞∩ n=1 ∞∪ m=n Am are called limes inferior and limes superior, respectively, of the sequence (An)n∈N.

Theorem 1.1(**Intersection of classes of sets**):Let I be an arbitrary index set, and assume that Ai is a σ-algebra for every i ∈ I. Hence the inters- ection AI := {A ⊂ Ω : A ∈ Ai for every i ∈ I} = ∩ i∈I Ai is a σ-algebra. The analogous statement holds for rings, σ-rings, algebras and λ- systems. However, it fails for semirings

Theorem 1.2 (**Generated σ-algebra**):Let E ⊂ 2Ω . Then there exists a smallest σ-algebra σ(E) with E ⊂ σ(E): σ(E) := ∩ A⊂2Ω is a σ-algebra A⊃E A.

Theorem 1.3(**∩-closed λ-system**):Let D ⊂ 2Ω be a λ-system. Then D is a π-system ⇐⇒ D is a σ-algebra.

Theorem 1.4(**Dynkin's π-λ theorem**): If E ⊂ 2Ω is a π-system, then σ(E) = δ(E).

Definition 1.7(**Topology**): Let Ω = ∅ be an arbitrary set. A class of sets τ ⊂ Ω is called a topology on Ω if it has the following three properties: (i) ∅, Ω ∈ τ . (ii) A ∩ B ∈ τ for any A, B ∈ τ . (iii) (∪ A∈F A) ∈ τ for any F ⊂ τ . The pair (Ω, τ) is called a **topological space**. The sets A ∈ τ are called open, and the sets A ⊂ Ω with Ac ∈ τ are called closed.

Definition 1.8(**Borel σ-algebra**):Let (Ω, τ) be a topological space. The σ- algebra B(Ω) := B(Ω, τ) := σ(τ) that is generated by the open sets is called the Borel σ-algebra on Ω. The elements A ∈ B(Ω, τ) are called Borel sets or Borel measurable sets.

Definition 1.8 (**Trace of a class of sets**):Let A ⊂ 2Ω be an arbitrary class of subsets of Ω and let A ∈ 2Ω \ {∅}. The class A| |A := {A ∩ B : B ∈ A} ⊂ 2A is called the trace of A on A or the restriction of A to A.

Set Functions

Set Functions (cont)

σ-finite	if there exists a sequence of sets Ω1, Ω2, . . . ∈ A such that Ω = ∞∪ n=1 Ωn and such that μ(Ωn) < ∞ for all n ∈ N.
Examples	
Dirac measure	Let ω ∈ Ω and δω (A) = 1A(ω) . Then δω is a probability measure on any σ-algebra A ⊂ 2Ω . δω is called the
uniform distri- bution	Let Ω be a finite nonempty set. By μ(A) := #A #Ω for A ⊂ Ω, we define a probability measure on A = 2Ω

Let A ⊂ 2Ω and let μ : A → [0, ∞] be a set function. We say that μ is	
monoton	μ(A) ≤ μ(B) for any two sets A, B ∈ A with A ⊂ B,
additiv	if μ (n⋃ i=1 Ai) = n∑ i=1 μ(Ai) for any choice of finitely many mutually disjoint sets A1, . . . , An ∈ A with n∪ i=1 Ai ∈ A,
σ-additive	if μ (∞⋃ i=1 Ai) = ∞∑ i=1 μ(Ai) for any choice of countably many mu- tually disjoint sets A1, A2, . . . ∈ A with ∞∪ i=1 Ai ∈ A,
subadditive	if for any choice of finitely many sets A, A1, . . . , An ∈ A with A ⊂ n∪ i=1 Ai, we have μ(A) ≤ n∑ i=1 μ(Ai), and
σ-subadditive	if for any choice of countably many sets A, A1, A2, . . . ∈ A with A ⊂ ∞∪ i=1 Ai, we have μ(A) ≤ ∞∑ i=1 μ(Ai)
Let A be a semiring and let μ : A → [0, ∞] be a set function with μ(∅) = 0. μ is called a	
content	if μ is additive,
premeasure	if μ is σ-additive,
measure	if μ is a premeasure and A is a σ-algebra
probability measure	if μ is a measure and μ(Ω) = 1
Let A be a semiring. A content μ on A is called	
finite	if μ(A) < ∞ for every A ∈ A