

### Linear Systems

Solution	Can have <b>no solution, one solution</b> or <b>infinitely many</b> . The solution is the <b>intersection</b>
Solution set	The set of all possible solutions
Consistency	A system is <b>consistent</b> if there is at least one solution otherwise it is <b>inconsistent</b>
Equivalent	Linear systems are <b>equivalent</b> if they have the same <b>solution set</b>
Row operations	<b>Replacement, interchange</b> and <b>scaling</b>
Row equivalent	If there is a sequence of <b>row operations</b> between two linear systems then the systems are <b>row equivalent</b> . Systems that are row equivalent has the same <b>solution set</b> .
Existence	If a system has a solution (i.e. <b>consistent</b> )
Uniqueness	Is the solution unique
Homogenous	A system is <b>homogenous</b> if it can be written in the form $Ax = 0$
Trivial solution	If a system only has a the solution $x = 0$ . A system with no <b>free variable</b> only have the trivial solution.
Non-trivial solution	A <b>nonzero</b> vector that satisfies $Ax = 0$ . Has free variable.

### Inverser of a Matrix

$C$  is invertible if  $CA = I^n$  and  $AC = I^n$

If  $A$  is  $(2 \times 2)$  then,  $A^{-1} =$

$$(A^{-1})^{-1} = A$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(A^T)^{-1} = (A^{-1})^T$$

### Linear Transformations

Transformation/mapping	$T(x)$ from $R^n$ to $R^m$
Image	For $x$ in $R^n$ the vector $T(x)$ in $R^m$ is called the image
Range	The set of all <b>images</b> of the vectors in the <b>domain</b> of $T(x)$
Criterion for a transformation to be linear	1. $T(u + v) = T(u) + T(v)$ 2. $T(cU) = cT(U)$
Standard Matrix	The matrix $A$ for a linear transformation $T$ , that satisfies $T(x) = Ax$ for all $x$ in $R^n$
Onto	A mapping $T$ is said to be <b>onto</b> if each $b$ in the codomain is the image of at least one $x$ in the domain. Range = Codomain. Solution existence. Col $A$ must match codomain.
One-to-one	If each $b$ in the <b>codomain</b> is only the image <b>at most</b> one $x$ in the <b>domain</b> . Solution Uniqueness.  $T$ is one-to-one if and only if the cols of $A$ are <b>linearly independent</b>
Free variable?	If the system has a free variable, then the system is <b>not</b> one-to-one. I.e. the homogenous system only has the trivial solution
Pivot in every row?	Then $T$ is <b>onto</b>

### Linear Transformations (cont)

Pivot in every column?	Then $T$ is <b>one-to-one</b>
To determine whether a vector $c$ is in the range of a $T$ . <b>Solution:</b> Let $T(x) = Ax$ . Solve the matrix equation $Ax = c$ . If the system is <b>consistent</b> , then $c$ is in the range of $T$ .	

### The Invertible Matrix Theorem

The following statements are equivalent i.e. **either they are all true or all false**. Let  $A$  be a  $(n \times n)$  matrix

$A$  is an invertible matrix.

$A$  is row equivalent to the  $n \times n$  identity matrix

$A$  has  $n$  pivot positions.

The equation  $Ax = 0$  has only the trivial solution

The columns of  $A$  form a linearly independent set

The linear transformation  $x \mapsto Ax$  is one-to-one.

The equation  $Ax = b$  has at least one solution for each  $b$  in  $R^n$

The columns of  $A$  span  $R^n$

The linear transformation  $x \mapsto Ax$  maps  $R^n$  onto  $R^n$

There is an  $n \times n$  matrix  $C$  such that  $\square$   
 $\square CA = I$ .

There is an  $n \times n$  matrix  $D$  such that  $\square$   
 $\square DA = I$ .

$A^T$  is an invertible matrix.

The columns of  $A$  form a basis of  $R^n$

$\text{Col } A = R^n$

$\text{Dim Col } A = n$

$\text{Rank } A = n$

$\text{Nul } A = \{0\}$

$\text{Dim Nul } A = 0$

The number 0 is not an eigenvalue of  $A$

The determinant of  $A$  is not 0



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### Elementary Matrices

**Elementary Matrix** Is obtained by performing a single elementary row operation on an **identity matrix**

Each elementary matrix **E** is **invertible**

A  $n \times n$  matrix **A** is invertible if and only if **A** is row equivalent to  $I^n$ .

$$A = E^{-1}I^n \text{ and } A^{-1} = EI^n = I^n$$

Row reduce the augmented matrix  $[A \ I]$  to  $[I \ A^{-1}]$

**NOTE** If **A** is not row equivalent to **I** then **A** is not invertible

### Linear Independence

A set of vectors are **linearly independent** if they cannot be created by any linear combinations of earlier vectors in the set.

If a set of vectors are **linear independent**, then the solution is **unique**

If the vector equation  $c_1v_1 + c_2v_2 + \dots + c_p v_p = 0$  only has a **trivial solution** the set of vectors are **linearly independent**

**Theorem:** If a set contains more vectors than there are entries in each vector, then the set is **linearly dependent**

**Theorem:** If a set of vectors contain the zero vector, then the set is **linearly dependent**

### Algebraic properties of a matrix

### Algebraic properties of a matrix (cont)

$$(AB)^T = B^T A^T$$

$$\text{For any scalar } r, (rA)^T = rA^T$$

### LU Factorization

**Factorization** of a matrix **A** is an equation that expresses **A** as a **product** of two or more matrices:

$$\text{Synthesis: } BC = A$$

$$\text{Analysis: } A = BC$$

**Assumption:** **A** is a  $m \times n$  matrix that can be row reduced without **interchanges**

**L:** is a  $m \times m$  unit lower triangular with 1's on the diagonal

**U:** is a  $m \times n$  echelon form of **A**

$$\text{U is equal to } E^*A = U, \text{ why } A = E^{-1}U = LU \text{ where } L = E^{-1}$$

See figure \*\* for how to find **L** and **U**

Find **x** by first solving  $Ly = b$  and then solving  $Ux = y$

### Row Reduction and Echelon forms

**Leading entry** A leading entry refers to the leftmost **non-zero** entry in a row

**Echelon form** Row equivalent systems can be reduced into **several different** echelon forms

**Reduced echelon form** A system is only row equivalent to **one** REF

**Forward phase** Reducing an augmented matrix **A** into an **echelon form**

**Backward phase** Reducing an augmented matrix **A** into a **reduced echelon form**

**Basic variables** Variables in **pivot columns**.

**Free variables** Variables that are not in **pivot columns**. When a system has a free variable the system is **consistent** but not **unique**

### Subspaces of $R^n$

A **subspace** of  $R^n$  is any set **H** in  $R^n$  that has three properties:

- The zero vector is in **H**
- For each **u** and **v** in **H**, the sum **u + v** is in **H**
- For each **u** in **H** and each scalar **c**, the vector **cu** is in **H**

**Zero subspace** is the set containing only the **zero vector** in  $R^n$

**Column space** is the set of all linear combinations of the columns of **A**.

**Null space (Nul A)** is the set of all solutions of the equation  $Ax = 0$

**Basis** for a subspace **H** is the set of **linearly independent** vectors that span **H**

In general, the **pivot columns** of **A** form a basis for **col A**

The number of vectors in any basis is **unique**. We call this number **dimension**

The **rank** of a matrix **A**, denoted by **rank A**, is the **dimension** of the **column space** of **A**

Determine whether **b** is in the **col A**.

**Solution:** **b** is only in **col A** if the equation  $Ax = b$  has a solution

Matrix and vector sum	$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$
Matrix, vector and scalar	$A(c\mathbf{u}) = c(A\mathbf{u})$
Associative law	$A(BC) = (AB)C$
Left distributive law	$A(B + C) = AB + AC$
Right distributive law	$(B + C)A = BA + CA$
Scalar multiplication	$r(AB) = (rA)B = A(rB)$
Identity matrix multi	$I^m A = A = A I^n$
Commute	If $AB = BA$ then we say that A and B <b>commute</b> with each others
	$(A^T)^T = A$
	$(A + B)^T = A^T + B^T$



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### Algebraic properties of a vector

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

$$\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u}$$

$$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$$

$$c(d\mathbf{u}) = (cd)\mathbf{u}$$



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