

Linear Algebra - MATH 232 Cheat Sheet by fionaw via cheatography.com/124375/cs/23750/

Basic Equations

Network Flows

- 1. the flow in an arc is only in one directions
- 2. flow into a node = flow out of a node
- 3. flow into the network = flow out of the network

Balancing Chemical Equations

- 1. add x's before each combo and both side
- 2. carbo = x1 + 2(x3), set as system, solve

Matrix

augmented	variables and soluti-
matrix	on(rhs)
coefficient	coefficients only, no rhs
matrix	

Vectors, Norm, Dot Product

Vectors, North, Dot Floude	Vectors, North, Dot Froduct		
maginitude (norm) of vecto	v = v		
if k>0, kv same direction as v	magnitude = k v		
if k<0, kv opposite direction to v	magnitude = k v		
vectors in R ⁿ (n = dimension)	v = (v1, v2,, vn)		
v = P1P2 = OP2 - OP1	displacement vector		
norm/magnitude of vector	sqrt((v1) ² + (v2) ²)		
v = 0 iff $v = 0$	kv = k v		
unit vector u in same direct as v	u = (1/ v) v		
e1 = (1,0) en = (0,1) in R ⁿ	standard unit vector		

Vectors, Norm, Dot Product (cont)

u·v = u1v1 + u2v2 dot product
...+unvn
||u|| ||v|| cos(θ)

u and v are orthogonal if u·v = 0 (cos(θ) = 0)
a set of vectors is an orthogonal set iff vi·vj
= 0,if i≠i

a set of vectors is an orthonormal set iff vi·vj = 0,if i \neq j, and ||vi|| = 1 for all i

 $(\mathbf{u} \cdot \mathbf{v})^2 \le ||\mathbf{u}||^2 ||\mathbf{v}||^2$ or Cauchy-Schwarz $|\mathbf{u} \cdot \mathbf{v}| \le ||\mathbf{u}|| ||\mathbf{v}||$ Inequality $\mathbf{d}(\mathbf{u}\mathbf{v}) \le \mathbf{d}(\mathbf{u}, \mathbf{w}) + \mathbf{d}(\mathbf{w}, \mathbf{v})$ Triangle Inequality $||\mathbf{u} + \mathbf{v}|| \le ||\mathbf{u}|| + ||\mathbf{v}||$

||v1 + v2 ... + vk|| = ||v1|| + ||v2|| ... + ||vk||

Lines and Planes

a vector equation with	x = x0 + tv,
parameter t	$-\infty < t < +\infty$
solutin set for 3 dimension line a plane	ear equation is
if x is a point on this plane (point-normal equation)	$n \cdot (x - x0) = 0$
A(x-x0)+B(y-y0)+C(z-z0) = 0	x0 = (x0,y0,z0),

n = (A, B, C)

Ax+By+Cz =

two planes are parallel if n1 = kn2, orthogonal if $n1 \cdot n2 = 0$

general/algebraic equation

Matrix Algebra, Identity and Inverse Matrix

(A + B)ij = (A)ij + (B)ij (A - B)ij = (A)ij - (B)ij (cA)ij = c(A)ij $(A^{T})ij = (A)ji$ (AB)ij = ai1b1j + ai2b2j + ... aikbkj

Inner Product (number) is $\mathbf{u}^{\mathsf{T}}\mathbf{v} = \mathbf{u} \cdot \mathbf{v}$, \mathbf{u} and \mathbf{v} same size

Outer Product (matrix) is uv^T , u and v can be any size

 $(A^{T})^{T} = A$ $(kA)^{T} = k(A)^{T}$ $(A+B)^{T} = A^{T} + B^{T}$ $(AB)^{T} = B^{T}A^{T}$ $tr(A^{T}) = tr(A)$ tr(AB) = tr(BA) $u^{T}v = tr(uv^{T})$ $tr(uv^{T}) = tr(vu^{T})$ $tr(A) = a11 + a22 ... + (A^{T})ij = Aji$

Identity matrix is square matrix with 1 along diagonals

If A is m x n, $A \square n = A$ and $\square mA = A$

a square matrix is $AB = \Box = BA$ invertible(nonsingular) if:

B is the inverse of A $B = A^{-1}$

if A has no inverse, A is not invertible (singular)

det(A) = ad - bc ≠ 0 is invertible

if A is invertible: $(AB)^{-1} = B^{-1}A^{-1}$ $(A^{n})^{-1} = A^{-n} = (A^{-1})^{n}$ $(A^{T})^{-1} = (A^{-1})^{T}$ $(kA)^{-1}$ $1/k(A^{-1}), k \neq 0$

Elementary Matrix and Unifying Theorem

elementary matrices are invertible

 $A^{-1} = Ek Ek-1 ... E2 E1$

 $[A | \Box] \rightarrow [\Box | A^{-1}]$

(how to find inverse of A)

 $Ax = b; x = A^{-1}b$



= ||u-v||

d(u,v) = 0 iff u = v

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 $d(u,v) = sqrt((u1-v1)^2 + (u2-v2)^2 ... (un-vn)^2)$

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Elementary Matrix and Unifying Theorem

- A -> RREF = □
- A can be express as a product of E
- A is invertible
- Ax = 0 has only the trivial solution
- Ax = b is consistent for every vector b in R^n
- Ax = b has eactly 1 solution for every b in
- colum and rowvectors of A are linealy independent
- $-\det(A) \neq 0$
- λ = 0 is not an eigenvalue of A
- TA is one to one and onto If not, then all no.

Consistency

 $EAx = Eb \rightarrow Rx = b'$, where b' = Eb

(Ax=b) [A|b] -> [EA|Eb](Rx = b')(but treat b as unknown: b1, b2...)

For it to be consistent, if R has zero rows at the bottom, b' that row must equal to zero

Homogeneous Systems

Linear Combination of the vectors:

v = c1v1 + c2v2 ... + cnvn

(use matrix to find c)

Ax = 0	Homogeneous
Ax = b	Non-homog- enous
x = x0 + t1v1 + t2v2 + tkvk	Homogeneous
x = t1v1 + t2v2 + tkvk	Non-homog- eneous
xp is any solution of NH system	x = xp + xh
and xh is a solution of H system	

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Examples of Subspaces

IF: w1, w2 are then w1+w2 are within S within S

- the zero vector 0 it self is a subspace

and kw1 is within S

- Rⁿ is a subspace of all vectors
- Lines and planes through the origin are subspaces
- The set of all vectors b such that Ax = b is consistent, is a subspace
- If {v1, v2, ...vk} is any set of vectors in Rⁿ, then the set W of all linear combinations of these vector is a subspace

 $W = \{c1v1 + c2v2 + ... ckvk\}$; c are within real numbers

Span

- the span of a set of vectors { v1, v2, ... vk} is the set of all linear combinations of these vectors

span { v1, v2, ... vk} = { v11t, t2v2, ..., tkvk}

If $S = \{ v1, v2, ... vk \}$, then W = span(S) is a subspace

Ax = b is consistent if and only if b is a linear combination of col(A)

Linear Independent

- if unique solution for a set of vectors, then it is linearly independent

c1v1 + c2v2 ... + cnvn = 0; all the c = 0

- for dependent, not all the c = 0

Dependent if:

- a linear combination of the other vectors
- a scalar multiple of the other
- a set of more than n vectors in Rⁿ

Independent if:

- the span of these two vectors form a plane

Linear Independent (cont)

- list the vectors as the columns of a matrix, row reduce it, if many solution, then it is dependent
- after RREF, the columns with leading 1's are a maxmially linearly independent subset according to Pivot Theorem

Diagonal, Triangular, Symmetric Matrices Diagonal all zeros along the Matrices diagonal zeros above diagonal Lower Triangular Upper zeros below the Triangular diagonal $A^T = A$ Symmetric if: $A^T = -A$ Skew-Symm-

Determinants	
det(A) = a1jC1j + a2jC2j + anjCnj	expansion along jth column
det(A) = ai1Ci1 + ai2Ci2 + ainCin	expansion along the ith row
Cij = (-1) ^{i+j} Mij	

etric if:

Mij = deleted ith row and jth column matrix

- pick the one with most zeros to calculate easier

$det(A^T) = det(A)$	$det(A^{-1}) =$	
	1/det(A)	
det(AB) = det(A)det(B)	$det(kA) = k^n det(A)$	
- A is invertible iff det(A) not equal 0		

- det of triangular or diagonal matrix is the product of the diagonal entries

det(A) for 2x2 matrix ad - bc

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Adjoint and Cramer's Rule $adj(A) = C^{T}$

C^T = matrix confactor

of A

 $A^{-1} = (1/det(A))$ adj(A)A = det(A) I adj(A)

auj(A)

x1 = det(A1) / x2 = det(A2) / det(A)

det(A)

xn = det(An) / det(A) not equal 0 det(A)

An is the

An is the matrix when the nth column is replaced by b

Hyperplane, Area/Volume

a hyperplane in $a1x1 + a2x2 \dots + anxn = R^n$ b

- can also written as ax = b

to find a^{perp} ax = 0, find the span

if A is 2x2 matrix:

- |det(A)| is the area of parallelogram

if A is 3x3 matrix:

- |det(A)| is the **volume** of parallelepiped
- subtract points to get three vectors, then make it to a matrix to find the area/volume

Cross Product

u x v = (u2v3 - u3v2, u3v1 - u1v3, u1v2 - u2v1)

 $u \times v = -v \times k(u \times v) = (ku) \times v = u \times (kv)$

u x u = 0 parallel vectors has 0 for

 $u (u \times v) = 0$ $v (u \times v) = 0$

u x v is perpendicular to span {u, v}

c.p.

 $||u \times v|| = ||u|| ||v|| \sin(\text{theta})$, where theta is the angle between vectors

Complex Number

complex number a +

(a + ib) + (c + id) = (a + c) + i(b + d)

(a + ib) - (c + id) = (a - c) + i(b - d)

(a + ib) (c + id) = (ac + bd) + i(ad + bc)

 $(a + bx) (c + dx) = (ac + bdx^2) + x(ad + bc)$

 $i^2 = -1$

z = a + ib z bar = a - ibthe length(magnitude) of $|z| = sqrt(z \times z)$

the length(magnitude) of vector z

 $= \operatorname{sqrt}(a^2 + b^2)$

 $z^{-1} = 1/z = z bar / |z|^2$

 $z1/z2 = z1z2^{-1}$

 $z = |z| (\cos(\theta) + i (\sin(\theta)))$ polar form (r = |z|)

 $z1z2 = |z1| |z2| (\cos(\theta 1 + \theta 2) + i (\sin(\theta 1 + \theta 2))$

 $z1/z2 = |z1| / |z2| (\cos(\theta 1 - \theta 2) + i (\sin(\theta 1 - \theta 2))$

 $z^{n} = r^{n}(\cos(n \theta) + i \sin(n r = |z|\theta))$

 $e^{i \text{ theta}} = \cos(\theta) + i \sin(\theta)$

 $e^{i pi} = -1$ $e^{i pi} + 1 = 0$ $z_1z_2 = r_1r_2 e^{i (\theta_1 + \theta_2)}$ $z_1^n = r_1^n e^{i n\theta}$

 $z1/z2 = r1/r2 e^{i(\theta 1 - \theta 2)}$

Eigenvalues and Eigenvectors

 $Ax = \lambda x$

 $\det(\lambda I - A) = (-1)^n \det(A - \lambda I)$

 $pa(\lambda) = 3x3$: $det(A - \lambda I)$; 2x2: $det(\lambda I - A)$

- solve for $(\lambda I - A)x = 0$ for eigenvectors

Work Flow:

- form matrix
- compute $pa(\lambda) = det(\lambda I A)$
- find roots of $pa(\lambda)$ -> eigenvalues of A
- plug in roots then solve for the equation

Linear Transformation

f: Rⁿ -> R^m, n = domain, m = co-domain

f(x1, x2, ...xn) = (y1, ...ym)

T: Rⁿ -> R^m is a linear transformatin if

1. T(cu) = cT(u)

2. T(u + v) = T(u) + T(v)

for any linear transformation, T(0) = 0

 $R\theta = [T(e1) T(e2)] = [\cos\theta]$ matrix for $-\sin\theta$] rotation

[sinθ

cos₀]

reflection across y-axis: T(x, y) = (-x, y)

reflection across x-axis: T(x, y) = (y, -x)

reflection across diagonal y = x, T(x, y) = (y, y)

orthogonal projection onto the x-axis: T(x, y) = (x, 0)

orthogonal projection onto the y-axis: T(x, y) = (0, y)

u = (1/||v||)v; express it vertically as u1 and u2

 $A = [(u1)^2 u2u1]$ projection $[u1u2 (u2)^2]$ matrix

contraction with $0 \le k < 1$ (shrink), k > 1 (stretch)

 $[x, y] \rightarrow [kx, ky]$

compression in x-direction [x, y] -> [kx, y]

compression in y-direction [x, y] -> [x, ky]

shear in x-direction T(x,y) = (x+ky, y);

[x+ky (1, k), y(0, 1)]

shear in y-direction T(x,y) = (x, y+kx);

[x (1, 0), y (k, 1)]

orthogonal projection on the xy-plane: [x, y, 0]

orthogonal projection on the xz-plane: [x, 0, z]

orthogonal projection on the yz-plane: [0, y , z]

reflection about the xy-plane: [x, y, -z]

reflection about the xz-plane: [x, -y, z]

reflection about the yz-plane: [-x, y, z]

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Orthogonal Transformation

an orthogonal transformation is a linear transformation T; $R^n \rightarrow R^n$ that preserves lengths; ||T(u)|| = ||u||

 $||T(u)|| = ||u|| \le T(x) \cdot T(y) = x \cdot y$ for all x,y in \mathbb{R}^n

orthogonal matrix is square matrix A such that $A^T = A^{-1}$

- 1. if A is orthogonal, then so is A^T and A⁻¹
- 2. a product of orthonal matrices is orthogonal
- 3. if A is orthogonal, then det(A) = 1 or -1
- 4. if A is orthogonal, then rows and columns of A are each orthonormal sets of vectors

Kernel, Range, Composition

ker(T) is the set of all vectors x such that T(x) = 0, RREF matrix, find the vector,

 $ker(T) = span\{(v)\}$

the solution space of Ax = 0 is the null space;

null(A) = ker(A)

range of T, ran(T) is the set of vectors y such that

y = T(x) for some x

ran(T) = col([T]) = span{ [col1], [col2] ...}; Ax

Important Facts:

- 1. T is one to one iff $ker(T) = \{0\}$
- 2. Ax = b, if consistent, has a unique solution

iff $null(A) = \{0\}$; Ax = 0 has only the trivial solution iff $null(A) = \{0\}$

Important facts 2:

- 1.T: $R^n \rightarrow R^m$ is onto iff the system Tx = y has a solution x in R^n for every y in R^m
- 2. Ax = b is consistent for every b in $R^{m}(A \text{ is onto})$ iff $col(A) = R^{m}$

The composition of T2 with T1 is: T2 ° T1

 $(T2 \circ T1)(x) = T2(T1(x)); T2 \circ T1: R^n -> R^m$

compostion of linear transformations corresponds to matrix application: [T2 ° T1] = [T1] [T2]

Kernel, Range, Composition (cont)

 $[T(\theta 1+\theta 2)] = [T\theta 2] \circ [T\theta 1];$

rotate then shear + shear then rotate

linear trans T: R^n -> R^m has an inverse iff T is one to one, T^{-1} : R^m -> R^n , $Tx = y <=> x = <math>T^{-1}v$

for Rn to Rn, $[T^{-1}] = [T]^{-1}$; $[T]^{-1} \circ T = 1n <=> [T^{-1}][T] = \square n$

1n is identity transformation; □n is identity matrix

Basis, Dimension, Rank

S is a basis for the subspace V of Rⁿ if: S is linearly idenpendent and span(S) = V

dim(V) = k, k is the # of vectors

row(A) = rows with leading ones after RREF

col(A) = columns with leading ones from original A

null(A) = free variable's vectors

 $null(A^{T})$ = after transform, the free variable vector

The Rank Theorem: $rank(A) = rank(A^{T})$ for any matrix have the same dimension

rank(A) = # of free vectors in span

dim(row(A)) = dim(col(A)) = rank(A)

dim(null(A)) = nullity(A)

Orthogonal Compliment, Dimention Theorem

 $S^\perp = \{v \in R^n \mid v \cdot w = 0 \text{ for all } w \in S\}$

 S^{\perp} is a subspace of R^{n} ; $S^{\perp} = span(S)^{\perp} = W^{\perp}$

 $\begin{aligned} \text{row}(\mathsf{A})^\perp &= \text{null}(\mathsf{A}) & \text{null}(\mathsf{A})^\perp &= \text{row}(\mathsf{A}) \\ & ((\mathsf{S}^\perp)^\perp &= \mathsf{S} \text{ iff } \mathsf{S} \text{ is} \\ & \text{subspace} \\ & \text{col}(\mathsf{A})^\perp &= \text{null}(\mathsf{A}^\mathsf{T}) & \text{null}(\mathsf{A}^\mathsf{T})^\perp &= \text{col}(\mathsf{A}) \end{aligned}$

The Dimension rank(A) + nullity(A) =Theorem n

A is m x n matrix (k + (n-k) = n)

 $\begin{array}{ll} \text{if W is a subspace} & \dim(W) + \dim(W^{\perp}) = \\ \text{of R}^n & n \end{array}$



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