# Linear Algebra - MATH 232 Cheat Sheet

by fionaw via cheatography.com/124375/cs/23750/

## Basic Equations

**Network Flows**

1. the flow in an arc is only in one directions
2. flow into a node = flow out of a node
3. flow into the network = flow out of the network

**Balancing Chemical Equations**

1. add x's before each combo and both side
2. carbo = x1 + 2(x3), set as system, solve

**Matrix**

| | |
|---|---|
| augmented matrix | variables and solution(rhs) |
| coefficient matrix | coefficients only, no rhs |

## Vectors, Norm, Dot Product

maginitude (norm) of vector v is $\|v\|$; $\|v\| \geq 0$

| | |
|---|---|
| if k>0, kv same direction as v | magnitude = $k\|v\|$ |
| if k<0, kv opposite direction to v | magnitude = $\|k\| \|v\|$ |
| vectors in $R^n$ (n = dimension) | v = (v1, v2, ..., vn) |
| v = P1P2 = OP2 - OP1 | displacement vector |
| norm/magnitude of vector **$\|v\|$** | **sqrt( $(v1)^2$+ $(v2)^2$...)** |
| $\|v\| = 0$ iff v =0 | $\|kv\| = \|k\| \|v\|$ |
| unit vector u in same direct as v | **u = (1/ $\|v\|$) v** |
| e1 = (1,0...) ... en = (0,...1) in $R^n$ | standard unit vector |

**d(u,v) = sqrt($(u1-v1)^2$ + $(u2-v2)^2$ ... $(un-vn)^2$) = $\|u-v\|$**

d(u,v) = 0 iff u = v

## Vectors, Norm, Dot Product (cont)

| | |
|---|---|
| **u·v = u1v1 + u2v2 ...+unvn** **$\|u\| \|v\| \cos(\theta)$** | dot product |

u and v are orthogonal if u·v = 0 (cos(θ) = 0)

a set of vectors is an orthogonal set iff vi·vj = 0,if i≠j

a set of vectors is an orthonormal set iff vi·vj = 0,if i≠j, and $\|vi\| = 1$ for all i

| | |
|---|---|
| **$(u\cdot v)^2 \leq \|u\|^2\|v\|^2$** or **$\|u\cdot v\| \leq \|u\| \|v\|$** | Cauchy-Schwarz Inequality |
| **d(uv) ≤ d(u,w) + d(w,v)** **$\|u+v\| \leq \|u\| + \|v\|$** | Triangle Inequality |

**$\|v1 + v2 ... + vk\| = \|v1\| + \|v2\| ... + \|vk\|$**

## Lines and Planes

| | |
|---|---|
| a vector equation with parameter t | x = x0 + tv, -∞ < t < +∞ |

solutin set for 3 dimension linear equation is a **plane**

| | |
|---|---|
| if x is a point on this plane **(point-normal equation)** | n·(x-x0) = 0 |
| **A(x-x0)+B(y-y0)+C(z-z0) = 0** | **x0 = (x0,y0,z0), n = (A, B, C)** |
| general/algebraic equation | Ax+By+Cz = D |

two planes are **parallel if n1 = kn2**, **orthogonal if n1·n2 = 0**

## Matrix Algebra, Identity and Inverse Matrix

| | |
|---|---|
| (A + B)ij = (A)ij + (B)ij | (A - B)ij = (A)ij - (B)ij |
| (cA)ij = c(A)ij | **$(A^T)$ij = (A)ji** |

(AB)ij = ai1b1j + ai2b2j + ... aikbkj

**Inner Product (number) is $u^Tv$ = u·v**, u and v same size

**Outer Product (matrix) is $uv^T$**, u and v can be any size

| | |
|---|---|
| $(A^T)^T = A$ | $(kA)^T = k(A)^T$ |
| $(A+B)^T = A^T + B^T$ | $(AB)^T = B^TA^T$ |
| $tr(A^T) = tr(A)$ | tr(AB) = tr(BA) |
| $u^Tv = tr(uv^T)$ | $tr(uv^T) = tr(vu^T)$ |
| **tr(A) = a11 + a22 ... + ann** | $(A^T)$ij = Aji |

Identity matrix is square matrix with 1 along diagonals

If A is m x n, A□n = A and □mA = A

| | |
|---|---|
| a square matrix is **invertible(nonsingular)** if: | **AB = □ = BA** |
| B is the inverse of A | $B = A^{-1}$ |

if A has no inverse, A is not invertible (singular)

**det(A) = ad - bc ≠ 0 is invertible**

| | |
|---|---|
| if A is invertible: | $(AB)^{-1} = B^{-1}A^{-1}$ |
| $(A^n)^{-1} = A^{-n} = (A^{-1})^n$ | $(A^T)^{-1} = (A^{-1})^T$ |
| $(kA)^{-1}$ | $1/k(A^{-1})$, k≠0 |

## Elementary Matrix and Unifying Theorem

elementary matrices are invertible

$A^{-1}$ = Ek Ek-1 ... E2 E1

**[ A | □ ] -> [ □ | $A^{-1}$ ]**
*(how to find inverse of A)*

Ax = b; **x = $A^{-1}$b**

## Elementary Matrix and Unifying Theorem (cont)

- A -> RREF = ☐
- A can be express as a product of E
- A is invertible
- Ax = 0 has only the trivial solution
- Ax = b is consistent for every vector b in $R^n$
- Ax = b has eactly 1 solution for every b in $R^n$
- colum and rowvectors of A are linealy independent
- det(A) ≠ 0
- λ = 0 is not an eigenvalue of A
- TA is one to one and onto

If not, then all no.

## Consistency

EAx = Eb -> **Rx = b' , where b' = Eb**

(Ax=b) **[ A | b ] -> [ EA | Eb ]** (Rx = b')
*(but treat b as unknown: b1, b2...)*

**For it to be consistent, if R has zero rows at the bottom, b' that row must equal to zero**

## Homogeneous Systems

Linear Combination of the vectors:
**v = c1v1 + c2v2 ... + cnvn**
*(use matrix to find c)*

| | |
|---|---|
| Ax = 0 | Homogeneous |
| Ax = b | Non-homogenous |
| **x = x0 + t1v1 + t2v2 ... + tkvk** | Homogeneous |
| **x = t1v1 + t2v2 ... + tkvk** | Non-homogeneous |
| xp is any solution of NH system<br>and xh is a solution of H system | **x = xp + xh** |

## Examples of Subspaces

| | |
|---|---|
| IF: w1, w2 are within S | then w1+w2 are within S<br>and kw1 is within S |

- the zero vector 0 it self is a subspace
- $R^n$ is a subspace of all vectors
- Lines and planes through the origin are subspaces
- The set of all vectors b such that Ax = b is consistent, is a subspace
- If {v1, v2, ...vk} is any set of vectors in $R^n$, then the set W of all linear combinations of these vector is a subspace

W = {c1v1 + c2v2 + ... ckvk}; c are within real numbers

## Span

- the span of a set of vectors { v1, v2, ... vk} is the set of all linear combinations of these vectors

span { v1, v2, ... vk} = { v11t, t2v2, ... , tkvk}

If S = { v1, v2, ... vk}, then W = span(S) is a subspace

Ax = b is consistent if and only if b is a linear combination of col(A)

## Linear Independent

- if unique solution for a set of vectors, then it is linearly independent

c1v1 + c2v2 ... + cnvn = 0; all the c = 0

- for dependent, not all the c = 0

Dependent if:
- a linear combination of the other vectors
- a scalar multiple of the other
- a set of more than n vectors in $R^n$

Independent if:
- the span of these two vectors form a plane

## Linear Independent (cont)

- list the vectors as the columns of a matrix, row reduce it, if many solution, then it is dependent
- after RREF, the columns with leading 1's are a maxmially linearly independent subset according to Pivot Theorem

## Diagonal, Triangular, Symmetric Matrices

| | |
|---|---|
| Diagonal Matrices | all zeros along the diagonal |
| Lower Triangular | zeros above diagonal |
| Upper Triangular | zeros below the diagonal |
| Symmetric if: | $\mathbf{A^T = A}$ |
| Skew-Symmetric if: | $\mathbf{A^T = -A}$ |

## Determinants

| | |
|---|---|
| **det(A) = a1jC1j + a2jC2j ... + anjCnj** | expansion along jth column |
| **det(A) = ai1Ci1 + ai2Ci2 ... + ainCin** | expansion along the ith row |

**Cij = $(-1)^{i+j}$ Mij**

Mij = deleted ith row and jth column matrix

- pick the one with most zeros to calculate easier

| | |
|---|---|
| det($A^T$) = det(A) | det($A^{-1}$) = 1/det(A) |
| det(AB) = det(A)det(B) | det(kA) = $k^n$det(A) |

- A is invertible iff det(A) not equal 0
- det of triangular or diagonal matrix is the product of the diagonal entries

| | |
|---|---|
| det(A) for 2x2 matrix | **ad - bc** |

## Adjoint and Cramer's Rule

| | |
|---|---|
| **adj(A) = $C^T$** | $C^T$ = matrix confactor of A |
| $A^{-1}$ = (1/det(A)) adj(A) | adj(A)A = det(A) I |
| **x1 = det(A1) / det(A)** | **x2 = det(A2) / det(A)** |
| **xn = det(An) / det(A)** | det(A) not equal 0 |
| An is the matrix when the nth column is replaced by b | |

## Hyperplane, Area/Volume

| | |
|---|---|
| a hyperplane in $R^n$ | **a1x1 + a2x2 ... + anxn = b** |
| - can also written as ax = b | |
| to find $a^{perp}$ | ax = 0, find the span |
| if A is 2x2 matrix:<br>- \|det(A)\| is the **area** of parallelogram | |
| if A is 3x3 matrix:<br>- \|det(A)\| is the **volume** of parallelepiped | |
| - subtract points to get three vectors, then make it to a matrix to find the area/volume | |

## Cross Product

| | |
|---|---|
| **u x v = (u2v3 - u3v2, u3v1 - u1v3, u1v2 - u2v1)** | |
| u x v = -v x u | k(u x v) = (ku) x v = u x (kv) |
| u x u = 0 | parallel vectors has 0 for c.p. |
| u (u x v) = 0 | v (u x v) = 0 |
| u x v is perpendicular to span {u, v} | |
| \|\|u x v\|\| = \|\|u\|\| \|\|v\|\| sin(theta), where theta is the angle between vectors | |

## Complex Number

| | |
|---|---|
| complex number | **a + ib** |
| **(a + ib) + (c + id) = (a + c) + i(b + d)** | |
| **(a + ib) - (c + id) = (a - c) + i(b - d)** | |
| **(a + ib) (c + id) = (ac + bd) + i(ad + bc)** | |
| **(a + bx) (c + dx) = (ac + bd$x^2$) + x(ad + bc)** | |
| $i^2$ = -1 | |
| **z = a + ib** | **z bar = a - ib** |
| the length(magnitude) of vector z | **\|z\|** = sqrt(z x z bar)<br>**= sqrt($a^2$ + $b^2$)** |
| **$z^{-1}$ = 1/z = z bar / $\|z\|^2$** | |
| **z1 / z2 = z1$z2^{-1}$** | |
| **z = \|z\| (cos(θ) + i (sin(θ))** | polar form (r = \|z\|) |
| **z1z2 = \|z1\| \|z2\| (cos(θ1 + θ2) + i (sin(θ1 + θ2))** | |
| **z1/z2 = \|z1\| / \|z2\| (cos(θ1 - θ2) + i (sin(θ1 - θ2))** | |
| $z^n$ = $r^n$(cos(n θ) + i sin(n θ)) | r = \|z\| |
| $e^{i\ theta}$ = cos(θ) + i sin(θ) | |
| $e^{i\ pi}$ = -1 | $e^{i\ pi}$ +1 = 0 |
| z1z2 = r1r2 $e^{i\ (θ1 + θ2)}$ | $z^n$ = $r^n$ $e^{i\ nθ}$ |
| z1 /z2 = r1 / r2 $e^{i\ (θ1 - θ2)}$ | |

## Eigenvalues and Eigenvectors

| |
|---|
| Ax= λx |
| **det(λI - A) = $(-1)^n$ det(A - λI)** |
| pa(λ) = 3x3: det(A - λI); 2x2: det(λI - A) |
| - solve for (λI - A)x = 0 for eigenvectors |
| **Work Flow:**<br>- form matrix<br>- compute pa(λ) = det(λI - A)<br>- find roots of pa(λ) -> eigenvalues of A<br>- plug in roots then solve for the equation |

## Linear Transformation

| | |
|---|---|
| **f: $R^n$ -> $R^m$, n = domain, m = co-domain**<br>f(x1, x2, ...xn) = (y1, ...ym) | |
| T: $R^n$ -> $R^m$ is a linear transformatin if<br>1. T(cu) = cT(u)<br>2. T(u +v) = T(u ) + T(v) | |
| for any linear transformation, T(0) = 0 | |
| **Rθ = [T(e1) T(e2)] = [cosθ −sinθ]<br>[sinθ cosθ]** | matrix for rotation |
| reflection across y-axis: T(x, y) = (-x, y) | |
| reflection across x-axis: T(x, y) = (y, -x) | |
| reflection across diagonal y = x, T(x, y) = (y, x) | |
| orthogonal projection onto the x-axis: T(x, y) = (x, 0) | |
| orthogonal projection onto the y-axis: T(x, y) = (0, y) | |
| u = (1/ \|\|v\|\|)v; express it vertically as u1 and u2 | |
| **A = [$(u1)^2$ u2u1]<br>[u1u2 $(u2)^2$]** | projection matrix |
| contraction with 0 ≤ k < 1 (shrink), k > 1 (stretch)<br>**[x, y] -> [kx, ky]** | |
| compression in x-direction **[x, y] -> [kx, y]** | |
| compression in y-direction **[x, y] -> [x, ky]** | |
| shear in x-direction T(x,y) = (x+ky, y);<br>**[x+ky (1, k), y( 0, 1)]** | |
| shear in y-direction T(x,y) = (x, y+kx);<br>**[x (1, 0), y (k, 1)]** | |
| orthogonal projection on the xy-plane: [x, y , 0] | |
| orthogonal projection on the xz-plane: [x, 0 , z] | |
| orthogonal projection on the yz-plane: [0, y , z] | |
| reflection about the xy-plane: [x, y, -z] | |
| reflection about the xz-plane: [x, -y, z] | |
| reflection about the yz-plane: [-x, y, z] | |

## Orthogonal Transformation

an orthogonal transformation is a linear transformation T; $R^n$ -> $R^n$ that preserves lengths; **||T(u)|| = ||u||**

**||T(u)|| = ||u|| <=> T(x)·T(y) = x·y for all x,y in $R^n$**

orthogonal matrix is square matrix A such that $\mathbf{A^T = A^{-1}}$

1. if A is orthogonal, then so is $A^T$ and $A^{-1}$

2. a product of orthonal matrices is orthogonal

3. if A is orthogonal, then det(A) = 1 or -1

4. if A is orthogonal, then rows and columns of A are each orthonormal sets of vectors

## Kernel, Range, Composition

ker(T) is the set of all vectors x such that T(x) = 0, RREF matrix, find the vector, **ker(T) = span{(v)}**

the solution space of Ax = 0 is the null space;
**null(A) = ker(A)**

range of T, ran(T) is the set of vectors y such that
y = T(x) for some x

ran(T) = col([T]) = span{ [col1], [col2] ...}; Ax = b

Important Facts:
1. T is one to one iff ker(T) = {0}
2. Ax = b, if consistent, has a unique solution
iff null(A) = {0}; Ax = 0 has only the trivial solution iff null(A) = {0}

Important facts 2:
1.T:$R^n$ -> $R^m$ is onto iff the system Tx = y has a solution x in $R^n$ for every y in $R^m$
2. Ax = b is consistent for every b in $R^m$(A is onto) iff col(A) = $R^m$

The composition of T2 with T1 is: T2 ∘ T1

(T2 ∘ T1)(x) = T2(T1(x)); T2 ∘ T1: $R^n$ -> $R^m$

compostion of linear transformations corresponds to matrix application: **[T2 ∘ T1] = [T1] [T2]**

## Kernel, Range, Composition (cont)

[T(θ1+θ2)] = [Tθ2] ∘ [Tθ1];
rotate then shear ≠ shear then rotate

linear trans T: $R^n$->$R^m$ has an inverse iff T is one to one, $T^{-1}$: $R^m$ -> $R^n$, **Tx = y <=> x = $\mathbf{T^{-1}y}$**

for Rn to Rn, $[T^{-1}] = [T]^{-1}$; $[T]^{-1}$∘T = 1n <=> $[T^{-1}]$[T]=☐n
1n is identity transformation; ☐n is identity matrix

## Basis, Dimension, Rank

S is a basis for the subspace V of $R^n$ if:
S is linearly idenpendent and span(S) = V

dim(V) = k, k is the # of vectors

row(A) = rows with leading ones after RREF

col(A) = columns with leading ones from original A

null(A) = free variable's vectors

null($A^T$) = after transform, the free variable vector

The Rank Theorem: rank(A) = rank($A^T$) for any matrix have the same dimension

rank(A) = # of free vectors in span

dim(row(A)) = dim(col(A)) = rank(A)

dim(null(A)) = nullity(A)

## Orthogonal Compliment, DImention Theorem

$S^\perp$ = {v ∈ $R^n$ | v · w = 0 for all w ∈ S}

$S^\perp$ is a subspace of $R^n$; $S^\perp$ = span(S)$^\perp$ = $W^\perp$

| | |
|---|---|
| row(A)$^\perp$ = null(A) | null(A)$^\perp$ = row(A)<br>((S$^\perp$)$^\perp$ = S iff S is subspace |
| col(A)$^\perp$ = null($A^T$) | null($A^T$)$^\perp$ = col(A) |
| The Dimension Theorem<br>A is m x n matrix | rank(A) + nullity(A) = n<br>(k + (n-k) = n) |
| if W is a subspace of $R^n$ | dim(W) + dim($W^\perp$) = n |