Basic Equations

Network Flows

1. the flow in an arc is only in one directions
2. flow into a node = flow out of a node
3. flow into the network = flow out of the network

Balancing Chemical Equations

1. add x's before each combo and both side
2. carbo = x1 + 2(x3), set as system, solve

Matrix

augmented matrix	variables and solution(rhs)
coefficient matrix	coefficients only, no rhs

Vectors, Norm, Dot Product

maginitude (norm) of vector v is ||v||; ||v|| ≥ 0

if k>0, kv same direction as v	magnitude = k||v||
if k<0, kv opposite direction to v	magnitude = |k| ||v||
vectors in R^n (n = dimension)	v = (v1, v2, ..., vn)
v = P1P2 = OP2 - OP1	displacement vector
norm/magnitude of vector **||v||**	**sqrt($(v1)^2$+($-v2)^2$...)**
||v|| = 0 iff v =0	||kv|| = |k| ||v||
unit vector u in same direct as v	**u = (1/ ||v||) v**
e1 = (1,0...) ... en = (0,...1) in R^n	standard unit vector

d(u,v) = sqrt($(u1-v1)^2$ + $(u2-v2)^2$ ... $(un-vn)^2$) = ||u-v||

d(u,v) = 0 iff u = v

Vectors, Norm, Dot Product (cont)

u·v = u1v1 + u2v2 ...+unvn **||u|| ||v|| cos(θ)**	dot product

u and v are orthogonal if u·v = 0 (cos(θ) = 0)

a set of vectors is an orthogonal set iff vi·vj = 0,if i≠j

a set of vectors is an orthonormal set iff vi·vj = 0,if i≠j, and ||vi|| = 1 for all i

$(u·v)^2$ ≤ $\|\|u\|\|^2\|\|v\|\|^2$ or **|u·v| ≤ ||u|| ||v||**	Cauchy-Schwarz Inequality
d(uv) ≤ d(u,w) + d(w,v) **||u+v|| ≤ ||u|| + ||v||**	Triangle Inequality

||v1 + v2 ... + vk|| = ||v1|| + ||v2|| ... + ||vk||

Lines and Planes

a vector equation with parameter t	x = x0 + tv, -∞ < t < +∞

solutin set for 3 dimension linear equation is a **plane**

if x is a point on this plane **(point-normal equation)**	n·(x-x0) = 0
A(x-x0)+B(y-y0)+C(z-z0) = 0	**x0 = (x0,y0,z0), n = (A, B, C)**
general/algebraic equation	Ax+By+Cz = D

two planes are **parallel if n1 = kn2, orthogonal if n1·n2 = 0**

Matrix Algebra, Identity and Inverse Matrix

(A + B)ij = (A)ij + (B)ij	(A - B)ij = (A)ij - (B)ij
(cA)ij = c(A)ij	**(A^T)ij = (A)ji**

(AB)ij = ai1b1j + ai2b2j + ... aikbkj

Inner Product (number) is $u^T v$ = u·v, u and v same size

Outer Product (matrix) is uv^T, u and v can be any size

$(A^T)^T = A$	$(kA)^T = k(A)^T$
$(A+B)^T = A^T + B^T$	$(AB)^T = B^T A^T$
tr(A^T) = tr(A)	tr(AB) = tr(BA)
$u^T v$ = tr(uv^T)	tr(uv^T) = tr(vu^T)
tr(A) = a11 + a22 ... + ann	(A^T)ij = Aji

Identity matrix is square matrix with 1 along diagonals

If A is m x n, A n = A and mA = A

a square matrix is **invertible(nonsingular)** if:	**AB = = BA**
B is the inverse of A	$B = A^{-1}$

if A has no inverse, A is not invertible (singular)

det(A) = ad - bc ≠ 0 is invertible

if A is invertible:	$(AB)^{-1} = B^{-1}A^{-1}$
$(A^n)^{-1} = A^{-n} = (A^{-1})^n$	$(A^T)^{-1} = (A^{-1})^T$
$(kA)^{-1}$	$1/k(A^{-1})$, k≠0

Elementary Matrix and Unifying Theorem

elementary matrices are invertible

A^{-1} = Ek Ek-1 **...** E2 E1

[A |] -> [| A^{-1}]
(how to find inverse of A)

Ax = b; **x = A^{-1}b**

Elementary Matrix and Unifying Theorem (cont)

- A -> RREF =
- A can be express as a product of E
- A is invertible
- Ax = 0 has only the trivial solution
- Ax = b is consistent for every vector b in R^n
- Ax = b has eactly 1 solution for every b in R^n
- colum and rowvectors of A are linealy independent
- det(A) ≠ 0
- λ = 0 is not an eigenvalue of A
- TA is one to one and onto

If not, then all no.

Consistency

EAx = Eb -> **Rx = b' , where b' = Eb**

(Ax=b) **[A | b] -> [EA | Eb]** (Rx = b')
(but treat b as unknown: b1, b2...)

For it to be consistent, if R has zero rows at the bottom, b' that row must equal to zero

Homogeneous Systems

Linear Combination of the vectors:
v = c1v1 + c2v2 ... + cnvn
(use matrix to find c)

Ax = 0	Homogeneous
Ax = b	Non-homog-enous
x = x0 + t1v1 + t2v2 ... + tkvk	Homogeneous
x = t1v1 + t2v2 ... + tkvk	Non-homog-eneous
xp is any solution of NH system and xh is a solution of H system	**x = xp + xh**

Examples of Subspaces

IF: w1, w2 are within S	then w1+w2 are within S and kw1 is within S

- the zero vector 0 it self is a subspace

- R^n is a subspace of all vectors

- Lines and planes through the origin are subspaces

- The set of all vectors b such that Ax = b is consistent, is a subspace

- If {v1, v2, ...vk} is any set of vectors in R^n, then the set W of all linear combinations of these vector is a subspace

W = {c1v1 + c2v2 + ... ckvk}; c are within real numbers

Span

- the span of a set of vectors { v1, v2, ... vk} is the set of all linear combinations of these vectors

span { v1, v2, ... vk} = { v11t, t2v2, ... , tkvk}

If S = { v1, v2, ... vk}, then W = span(S) is a subspace

Ax = b is consistent if and only if b is a linear combination of col(A)

Linear Independent

- if unique solution for a set of vectors, then it is linearly independent

c1v1 + c2v2 ... + cnvn = 0; all the c = 0

- for dependent, not all the c = 0

Dependent if:
- a linear combination of the other vectors
- a scalar multiple of the other
- a set of more than n vectors in R^n

Independent if:
- the span of these two vectors form a plane

Linear Independent (cont)

- list the vectors as the columns of a matrix, row reduce it, if many solution, then it is dependent

- after RREF, the columns with leading 1's are a maxmially linearly independent subset according to Pivot Theorem

Diagonal, Triangular, Symmetric Matrices

Diagonal Matrices	all zeros along the diagonal
Lower Triangular	zeros above diagonal
Upper Triangular	zeros below the diagonal
Symmetric if:	**$A^T = A$**
Skew-Symm-etric if:	**$A^T = -A$**

Determinants

det(A) = a1jC1j + a2jC2j ... + anjCnj	expansion along jth column
det(A) = ai1Ci1 + ai2Ci2 ... + ainCin	expansion along the ith row

$Cij = (-1)^{i+j}$ Mij

Mij = deleted ith row and jth column matrix

- pick the one with most zeros to calculate easier

$det(A^T) = det(A)$	$det(A^{-1}) = 1/det(A)$
det(AB) = det(A)det(B)	$det(kA) = k^n det(A)$

- A is invertible iff det(A) not equal 0

- det of triangular or diagonal matrix is the product of the diagonal entries

det(A) for 2x2 matrix	**ad - bc**

Adjoint and Cramer's Rule

adj(A) = C^T	C^T = matrix confactor of A
A^{-1} = (1/det(A)) adj(A)	adj(A)A = det(A) I
x1 = det(A1) / det(A)	**x2 = det(A2) / det(A)**
xn = det(An) / det(A)	det(A) not equal 0
An is the matrix when the nth column is replaced by b	

Hyperplane, Area/Volume

a hyperplane in R^n	**a1x1 + a2x2 ... + anxn = b**
- can also written as ax = b	
to find a^{perp}	ax = 0, find the span
if A is 2x2 matrix: - |det(A)| is the **area** of parallelogram	
if A is 3x3 matrix: - |det(A)| is the **volume** of parallelepiped	
- subtract points to get three vectors, then make it to a matrix to find the area/volume	

Cross Product

u x v = (u2v3 - u3v2, u3v1 - u1v3, u1v2 - u2v1)	
u x v = -v x u	k(u x v) = (ku) x v = u x (kv)
u x u = 0	parallel vectors has 0 for c.p.
u (u x v) = 0	v (u x v) = 0
u x v is perpendicular to span {u, v}	
||u x v|| = ||u|| ||v|| sin(theta), where theta is the angle between vectors	

Complex Number

complex number	**a + ib**
(a + ib) + (c + id) = (a + c) + i(b + d)	
(a + ib) - (c + id) = (a - c) + i(b - d)	
(a + ib) (c + id) = (ac + bd) + i(ad + bc)	
(a + bx) (c + dx) = (ac + bdx^2) + x(ad + bc)	
i^2 = -1	
z = a + ib	**z bar = a - ib**
the length(magnitude) of vector z	**|z|** = sqrt(z x z bar) = **sqrt(a^2 + b^2)**
z^{-1} = 1/z = z bar / $\|z\|^2$	
z1 / z2 = z1$z2^{-1}$	
z = |z| (cos(θ) + i (sin(θ))	polar form (r = |z|)
z1z2 = |z1| |z2| (cos(θ1 + θ2) + i (sin(θ1 + θ2))	
z1/z2 = |z1| / |z2| (cos(θ1 - θ2) + i (sin(θ1 - θ2))	
z^n = r^n(cos(n θ) + i sin(n θ))	r = |z|
$e^{i\ theta}$ = cos(θ) + i sin(θ)	
$e^{i\ pi}$ = -1	$e^{i\ pi}$ +1 = 0
z1z2 = r1r2 $e^{i\ (θ1 + θ2)}$	z^n = r^n $e^{i\ nθ}$
z1 /z2 = r1 / r2 $e^{i\ (θ1 - θ2)}$	

Eigenvalues and Eigenvectors

Ax= λx
det(λI - A) = $(-1)^n$ det(A - λI)
pa(λ) = 3x3: det(A - λI); 2x2: det(λI - A)
- solve for (λI - A)x = 0 for eigenvectors
Work Flow: - form matrix - compute pa(λ) = det(λI - A) - find roots of pa(λ) -> eigenvalues of A - plug in roots then solve for the equation

Linear Transformation

f: R^n -> R^m, n = domain, m = co-domain f(x1, x2, ...xn) = (y1, ...ym)	
T: R^n -> R^m is a linear transformatin if 1. T(cu) = cT(u) 2. T(u +v) = T(u + T(v)	
for any linear transformation, T(0) = 0	
Rθ = [T(e1) T(e2)] = [cosθ −sinθ] [sinθ cosθ]	matrix for rotation
reflection across y-axis: T(x, y) = (-x, y)	
reflection across x-axis: T(x, y) = (y, -x)	
reflection across diagonal y = x, T(x, y) = (y, x)	
orthogonal projection onto the x-axis: T(x, y) = (x, 0)	
orthogonal projection onto the y-axis: T(x, y) = (0, y)	
u = (1/ ||v||)v; express it vertically as u1 and u2	
A = [$(u1)^2$ u2u1] [u1u2 $(u2)^2$]	projection matrix
contraction with 0 ≤ k < 1 (shrink), k > 1 (stretch) **[x, y] -> [kx, ky]**	
compression in x-direction **[x, y] -> [kx, y]**	
compression in y-direction **[x, y] -> [x, ky]**	
shear in x-direction T(x,y) = (x+ky, y); **[x+ky (1, k), y(0, 1)]**	
shear in y-direction T(x,y) = (x, y+kx); **[x (1, 0), y (k, 1)]**	
orthogonal projection on the xy-plane: [x, y , 0]	
orthogonal projection on the xz-plane: [x, 0 , z]	
orthogonal projection on the yz-plane: [0, y , z]	
reflection about the xy-plane: [x, y, -z]	
reflection about the xz-plane: [x, -y, z]	
reflection about the yz-plane: [-x, y, z]	

Orthogonal Transformation

an orthogonal transformation is a linear transformation T; R^n -> R^n that preserves lengths; **||T(u)|| = ||u||**

||T(u)|| = ||u|| <=> T(x)·T(y) = x·y for all x,y in R^n

orthogonal matrix is square matrix A such that **$A^T = A^{-1}$**

1. if A is orthogonal, then so is A^T and A^{-1}

2. a product of orthonal matrices is orthogonal

3. if A is orthogonal, then det(A) = 1 or -1

4. if A is orthogonal, then rows and columns of A are each orthonormal sets of vectors

Kernel, Range, Composition

ker(T) is the set of all vectors x such that T(x) = 0, RREF matrix, find the vector, **ker(T) = span{(v)}**

the solution space of Ax = 0 is the null space;
null(A) = ker(A)

range of T, ran(T) is the set of vectors y such that
y = T(x) for some x

ran(T) = col([T]) = span{ [col1], [col2] ...}; Ax = b

Important Facts:
1. T is one to one iff ker(T) = {0}
2. Ax = b, if consistent, has a unique solution
iff null(A) = {0}; Ax = 0 has only the trivial solution iff null(A) = {0}

Important facts 2:
1.T:R^n -> R^m is onto iff the system Tx = y has a solution x in R^n for every y in R^m
2. Ax = b is consistent for every b in R^m(A is onto) iff col(A) = R^m

The composition of T2 with T1 is: T2 ∘ T1

(T2 ∘ T1)(x) = T2(T1(x)); T2 ∘ T1: R^n -> R^m

compostion of linear transformations corresponds to matrix application: **[T2 ∘ T1] = [T1][T2]**

Kernel, Range, Composition (cont)

[T(θ1+θ2)] = [Tθ2] ∘ [Tθ1];
rotate then shear ≠ shear then rotate

linear trans T: R^n->R^m has an inverse iff T is one to one, T^{-1}: R^m -> R^n, **Tx = y <=> x = T^{-1}y**

for Rn to Rn, $[T^{-1}] = [T]^{-1}$; $[T]^{-1}$∘T = 1n <=> $[T^{-1}]$[T]= n
1n is identity transformation; n is identity matrix

Basis, Dimension, Rank

S is a basis for the subspace V of R^n if:
S is linearly idenpendent and span(S) = V

dim(V) = k, k is the # of vectors

row(A) = rows with leading ones after RREF

col(A) = columns with leading ones from original A

null(A) = free variable's vectors

null(A^T) = after transform, the free variable vector

The Rank Theorem: rank(A) = rank(A^T) for any matrix have the same dimension

rank(A) = # of free vectors in span

dim(row(A)) = dim(col(A)) = rank(A)

dim(null(A)) = nullity(A)

Orthogonal Compliment, DImention Theorem

$S^\perp$ = {v ∈ R^n | v · w = 0 for all w ∈ S}

$S^\perp$ is a subspace of R^n; $S^\perp$ = span(S)$^\perp$ = $W^\perp$

row(A)$^\perp$ = null(A)	null(A)$^\perp$ = row(A) (($S^\perp$)$^\perp$ = S iff S is subspace
col(A)$^\perp$ = null(A^T)	null(A^T)$^\perp$ = col(A)
The Dimension Theorem A is m x n matrix	rank(A) + nullity(A) = n (k + (n-k) = n)
if W is a subspace of R^n	dim(W) + dim($W^\perp$) = n