

### Definition of Groups

**Binary Operation** Let  $G$  be a set. A binary operation on  $G$  is a function that assigns each ordered pair of elements of  $G$  an element of  $G$ .

**Group** Let  $G$  be a set together with a binary operation (usually called multiplication) that assigns to each ordered pair  $(a, b)$  of elements of  $G$  an element in  $G$  denoted by  $ab$ . We say  $G$  is a group under this operation if the following three properties are satisfied.

**Properties to Satisfy (Group)** 1. Closure 2. Associativity 3. Identity 4. Inverses

**Abelian group** If a group has the property that  $ab = ba$  for every pair of elements  $a$  and  $b$ , we say the group is Abelian.

**Associativity** The operation is associative; that is,  $(ab)c = a(bc)$  for all  $a, b, c$  in  $G$ .

### Definition of Groups (cont)

**Identity** There is an element  $e$  (called the identity) in  $G$  such that  $ae = ea = a$  for all  $a$  in  $G$ .

**Inverses** For each element  $a$  in  $G$ , there is an element  $b$  in  $G$  (called an inverse of  $a$ ) such that  $ab = ba = e$ .

**Modular Arithmetic** When  $a = qn + r$ , where  $q$  is the quotient and  $r$  is the remainder upon dividing  $a$  by  $n$ , we write  $a \bmod n = r$ .

### Elementary Properties of Groups

**Theorem 2.1 Uniqueness of the Identity** In a group  $G$ , there is only one identity element.

**Theorem 2.2 Cancellation** In a group  $G$ , the right and left cancellation laws hold; that is,  $ba = ca$  implies  $b = c$ , and  $ab = ac$  implies  $b = c$ .

**Theorem 2.3 Uniqueness of Inverses** For each element  $a$  in a group  $G$ , there is a unique element  $b$  in  $G$  such that  $ab = ba = e$ .

**Theorem 2.4 Socks--Shoes Property** For group elements  $a$  and  $b$ ,  $(ab)^{-1} = b^{-1}a^{-1}$ .

### Subgroup Tests

**One-Step Subgroup Test** Let  $G$  be a group and  $H$  a nonempty subset of  $G$ . If  $ab^{-1}$  is in  $H$  whenever  $a$  and  $b$  are in  $H$ , then  $H$  is a subgroup of  $G$ . (In additive notation, if  $a - b$  is in  $H$  whenever  $a$  and  $b$  are in  $H$ , then  $H$  is a subgroup of  $G$ .)

**Two-Step Subgroup Test** Let  $G$  be a group and let  $H$  be a nonempty subset of  $G$ . If  $ab$  is in  $H$  whenever  $a$  and  $b$  are in  $H$  ( $H$  is closed under the operation), and  $a^{-1}$  is in  $H$  whenever  $a$  is in  $H$  ( $H$  is closed under taking inverses), then  $H$  is a subgroup of  $G$ .

**Theorem 3.3 Finite Subgroup Test** Let  $H$  be a nonempty finite subset of a group  $G$ . If  $H$  is closed under the operation of  $G$ , then  $H$  is a subgroup of  $G$ .



### Examples of Subgroups

**Theorem 3.4**  $\langle a \rangle$  is a subgroup of  $G$ .

**Center of a Group** The center,  $Z(G)$ , of a group  $G$  is the subset of elements in  $G$  that commute with every element of  $G$ . In symbols,  
 $Z(G) = \{a \in G \mid ax = xa \text{ for all } x \text{ in } G\}$ .

**Theorem 3.5** The center of a group  $G$  is a subgroup of  $G$ .

**Center is a Subgroup**

**Centralizer of a in  $G$**  Let  $a$  be a fixed element of a group  $G$ . The centralizer of  $a$  in  $G$ ,  $C(a)$ , is the set of all elements in  $G$  that commute with  $a$ . In symbols,  
 $C(a) = \{g \in G \mid ga = ag\}$ .

**Theorem 3.6** For each  $a$  in a group  $G$ , the centralizer of  $a$  is a subgroup of  $G$ .

**Is a Subgroup**

### Terminology and Notation

**Order of a Group** The number of elements of a group (finite or infinite) is called its order. We will use  $|G|$  to denote the order of  $G$ .

### Terminology and Notation (cont)

**Order of an Element** The order of an element  $g$  in a group  $G$  is the smallest positive integer  $n$  such that  $g^n = e$ . (In additive notation, this would be  $ng = 0$ .) If no such integer exists, we say that  $g$  has infinite order. The order of an element  $g$  is denoted by  $|g|$ .

**Subgroup** If a subset  $H$  of a group  $G$  is itself a group under the operation of  $G$ , we say that  $H$  is a subgroup of  $G$ .



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