

Definition of Groups

Binary Operation Let G be a set. A binary operation on G is a function that assigns each ordered pair of elements of G an element of G .

Group Let G be a set together with a binary operation (usually called multiplication) that assigns to each ordered pair (a, b) of elements of G an element in G denoted by ab . We say G is a group under this operation if the following three properties are satisfied.

Properties to Satisfy (Group) 1. Closure 2. Associativity 3. Identity 4. Inverses

Abelian group If a group has the property that $ab = ba$ for every pair of elements a and b , we say the group is Abelian.

Associativity The operation is associative; that is, $(ab)c = a(bc)$ for all a, b, c in G .

Definition of Groups (cont)

Identity There is an element e (called the identity) in G such that $ae = ea = a$ for all a in G .

Inverses For each element a in G , there is an element b in G (called an inverse of a) such that $ab = ba = e$.

Modular Arithmetic When $a = qn + r$, where q is the quotient and r is the remainder upon dividing a by n , we write $a \bmod n = r$.

Elementary Properties of Groups

Theorem 2.1 Uniqueness of the Identity In a group G , there is only one identity element.

Theorem 2.2 Cancellation In a group G , the right and left cancellation laws hold; that is, $ba = ca$ implies $b = c$, and $ab = ac$ implies $b = c$.

Theorem 2.3 Uniqueness of Inverses For each element a in a group G , there is a unique element b in G such that $ab = ba = e$.

Theorem 2.4 Socks-Shoes Property For group elements a and b , $(ab)^{-1} = b^{-1}a^{-1}$.

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Subgroup Tests

One-Step Subgroup Test Let G be a group and H a nonempty subset of G . If ab^{-1} is in H whenever a and b are in H , then H is a subgroup of G . (In additive notation, if $a - b$ is in H whenever a and b are in H , then H is a subgroup of G .)

Two-Step Subgroup Test Let G be a group and let H be a nonempty subset of G . If ab is in H whenever a and b are in H (H is closed under the operation), and a^{-1} is in H whenever a is in H (H is closed under taking inverses), then H is a subgroup of G .

Theorem 3.3 Finite Subgroup Test Let H be a nonempty finite subset of a group G . If H is closed under the operation of G , then H is a subgroup of G .



Examples of Subgroups

Theorem 3.4 $\langle a \rangle$ is a subgroup of G .
 Let G be a group, and let a be any element of G . Then, $\langle a \rangle$ is a subgroup of G .

Center of a Group
 The center, $Z(G)$, of a group G is the subset of elements in G that commute with every element of G . In symbols,
 $Z(G) = \{a \in G \mid ax = xa \text{ for all } x \text{ in } G\}$.

Theorem 3.5
 The center of a group G is a subgroup of G .

Center is a Subgroup

Centralizer of a in G
 Let a be a fixed element of a group G . The centralizer of a in G , $C(a)$, is the set of all elements in G that commute with a . In symbols,
 $C(a) = \{g \in G \mid ga = ag\}$.

Theorem 3.6 $C(a)$ is a subgroup of G .
 For each a in a group G , the centralizer of a is a subgroup of G .

Is a Subgroup

Terminology and Notation

Order of a Group
 The number of elements of a group (finite or infinite) is called its order. We will use $|G|$ to denote the order of G .

Terminology and Notation (cont)

Order of an Element
 The order of an element g in a group G is the smallest positive integer n such that $g^n = e$. (In additive notation, this would be $ng = 0$.) If no such integer exists, we say that g has infinite order. The order of an element g is denoted by $|g|$.

Subgroup
 If a subset H of a group G is itself a group under the operation of G , we say that H is a subgroup of G .



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