

Circular Functions Definitions

Name	Right-Triangle Definition	Domain	Range
Sine Function	$\sin(\theta) = o/h$	$(-\infty, \infty)$	$[-1, 1]$
Cosine Function	$\cos(\theta) = a/h$	$(-\infty, \infty)$	$[-1, 1]$
Tangent Function	$\tan(\theta) = o/a$	$\{\theta \theta \neq \pi/2 \pm \pi n\}$	$(-\infty, \infty)$
Cosecant Function	$\csc(\theta) = h/o$	$\{\theta \theta \neq \pm \pi n\}$	$(-\infty, -1] \cup [1, \infty)$
Secant Function	$\sec(\theta) = h/a$	$\{\theta \theta \neq \pi/2 \pm \pi n\}$	$(-\infty, -1] \cup [1, \infty)$
Cotangent Function	$\cot(\theta) = a/o$	$\{\theta \theta \neq \pm \pi n\}$	$(-\infty, \infty)$
Inverse Sine Function	$\arcsin(o/h) = \theta$	$[-1, 1]$	$[-\pi/2, \pi/2]$
Inverse Cosine Function	$\arccos(a/h) = \theta$	$[-1, 1]$	$[0, \pi]$
Inverse Tangent Function	$\arctan(o/a) = \theta$	$(-\infty, \infty)$	$(-1, 1)$
Inverse Cosecant Function	$\operatorname{arccsc}(h/o) = \theta$	$(-\infty, -1) \cup (1, \infty)$	$[-\pi/2, 0) \cup (0, \pi/2]$
Inverse Secant Function	$\operatorname{arcsec}(h/a) = \theta$	$(-\infty, -1) \cup (1, \infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$
Inverse Cotangent Function	$\operatorname{arccot}(a/o) = \theta$	$(-\infty, \infty)$	$(0, 1)$
Circular Euler Relation	$e^{\pm i\theta} = \cos(\theta) \pm i\sin(\theta)$		
De Moivre's Theorem	$e^{in\theta} = (\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$		

$n \in \mathbb{N}_1 = \{1, 2, 3, 4, 5, \dots\}$

"h" is the "hypotenuse" leg of a right triangle. It is directly across the right (90°) angle, and it has the longest length of the three sides.

"o" is the "opposite" leg of a right triangle. It is directly across the chosen angle θ .

"a" is the "adjacent" leg of a right triangle. It is the leg that is neither the hypotenuse leg, nor the opposite leg.

By the Pythagorean theorem, $o^2 + a^2 = h^2$

Hyperbolic Functions Definitions

Name	Exponential Definition	Domain	Range
Hyperbolic Sine Function	$\sinh(\theta) = (e^\theta - e^{-\theta})/2$	$(-\infty, \infty)$	$(-\infty, \infty)$
Hyperbolic Cosine Function	$\cosh(\theta) = (e^\theta + e^{-\theta})/2$	$(-\infty, \infty)$	$[1, \infty)$
Hyperbolic Tangent Function	$\tanh(\theta) = (e^\theta - e^{-\theta})/(e^\theta + e^{-\theta})$	$(-\infty, \infty)$	$(-1, 1)$
Hyperbolic Cosecant Function	$\operatorname{csch}(\theta) = 2/(e^\theta - e^{-\theta})$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
Hyperbolic Secant Function	$\operatorname{sech}(\theta) = 2/(e^\theta + e^{-\theta})$	$(-\infty, \infty)$	$(0, 1]$



By **CROSSANT** (CROSSANT)
cheatography.com/crossant/

Published 26th August, 2024.
 Last updated 25th July, 2025.
 Page 1 of 9.

Sponsored by **CrosswordCheats.com**
 Learn to solve cryptic crosswords!
<http://crosswordcheats.com>

Hyperbolic Functions Definitions (cont)

Hyperbolic Cotangent Function	$\coth(\theta) = (e^\theta + e^{-\theta}) / (e^\theta - e^{-\theta})$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, -1) \cup (1, \infty)$
Inverse Hyperbolic Sine Function	$\operatorname{arcsinh}(x) = \ln(x + \sqrt{x^2 + 1})$	$(-\infty, \infty)$	$(-\infty, \infty)$
Inverse Hyperbolic Cosine Function	$\operatorname{arcosh}(x) = \ln(x + \sqrt{x^2 - 1})$	$[1, \infty)$	$[0, \infty)$
Inverse Hyperbolic Tangent Function	$\operatorname{artanh}(x) = \frac{1}{2} \ln((1+x)/(1-x))$	$(-1, 1)$	$(-\infty, \infty)$
Inverse Hyperbolic Cosecant Function	$\operatorname{arccsch}(x) = \ln((1 \pm \sqrt{1+x^2})/x)$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
Inverse Hyperbolic Secant Function	$\operatorname{arcsech}(x) = \ln((1 + \sqrt{1-x^2})/x)$	$(0, 1]$	$[0, \infty)$
Inverse Hyperbolic Cotangent Function	$\operatorname{arcoth}(x) = \frac{1}{2} \ln((x+1)/(x-1))$	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, 0) \cup (0, \infty)$
Hyperbolic Euler Relation	$e^{\pm\theta} = \cosh(\theta) \pm \sinh(\theta)$		
De Moivre's Theorem (Hyperbolic)	$e^{n\theta} = (\cosh(\theta) + \sinh(\theta))^n = \cosh(n\theta) + \sinh(n\theta)$		
$n \in \mathbb{N}_1 = \{1, 2, 3, 4, 5, \dots\}$			

Complex Definitions

Name	Complex Relation	Circular-Hyperbolic Relation
Complex Sine	$\sin(z) = (e^{iz} - e^{-iz})/2i$	$\sin(z) = -i \sinh(iz)$
Complex Cosine	$\cos(z) = (e^{iz} + e^{-iz})/2$	$\cos(z) = \cosh(iz)$
Complex Tangent	$\tan(z) = i(e^{iz} - e^{-iz})/(e^{iz} + e^{-iz})$	$\tan(z) = i \tanh(iz)$
Complex Cosecant	$\csc(z) = 2i/(e^{iz} - e^{-iz})$	$\csc(z) = i \operatorname{csch}(iz)$
Complex Secant	$\sec(z) = 2/(e^{iz} + e^{-iz})$	$\sec(z) = \operatorname{sech}(iz)$
Complex Cotangent	$\cot(z) = i(e^{iz} + e^{-iz})/(e^{iz} - e^{-iz})$	$\cot(z) = i \coth(iz)$
Complex Inverse Sine	$\arcsin(z) = -i \ln(iz \pm \sqrt{1-z^2})$	$\arcsin(z) = -i \operatorname{arcsinh}(iz)$
Complex Inverse Cosine	$\arccos(z) = -i \ln(z \pm i \sqrt{1-z^2})$	$\arccos(z) = \pm i \operatorname{arcosh}(z)$
Complex Inverse Tangent	$\arctan(z) = (i/2) \ln((i+z)/(i-z))$	$\arctan(z) = -i \operatorname{artanh}(iz)$
Complex Inverse Cosecant	$\operatorname{arccsc}(z) = -i \ln((i + \sqrt{z^2 - 1})/z)$	$\operatorname{arccsc}(z) = i \operatorname{arccsch}(iz)$
Complex Inverse Secant	$\operatorname{arcsec}(z) = -i \ln((1 + \sqrt{1-z^2})/z)$	$\operatorname{arcsec}(z) = \pm i \operatorname{arcsech}(z)$
Complex Inverse Cotangent	$\operatorname{arccot}(z) = -(i/2) \ln((z+i)/(z-i))$	$\operatorname{arccot}(z) = \pm i \operatorname{arcoth}(iz)$
Complex Hyperbolic Sine	None	$\sinh(z) = -i \sin(iz)$



Complex Definitions (cont)

Complex Hyperbolic Cosine	None	$\cosh(z) = \cos(iz)$
Complex Hyperbolic Tangent	None	$\tanh(z) = -i \tan(iz)$
Complex Hyperbolic Cosecant	None	$\operatorname{csch}(z) = i \operatorname{csc}(iz)$
Complex Hyperbolic Secant	None	$\operatorname{sech}(z) = \sec(iz)$
Complex Hyperbolic Cotangent	None	$\operatorname{coth}(z) = i \cot(iz)$
Complex Inverse Hyperbolic Sine	None	$\operatorname{arcsinh}(z) = -i \arcsin(iz)$
Complex Inverse Hyperbolic Cosine	None	$\operatorname{arccosh}(z) = \pm i \arccos(z)$
Complex Inverse Hyperbolic Tangent	None	$\operatorname{arctanh}(z) = -i \arctan(iz)$
Complex Inverse Hyperbolic Cosecant	None	$\operatorname{arccsch}(z) = -i \operatorname{arccsc}(iz)$
Complex Inverse Hyperbolic Secant	None	$\operatorname{arcsech}(z) = \pm i \operatorname{arcsec}(z)$
Complex Inverse Hyperbolic Cotangent	None	$\operatorname{arccoth}(z) = -i \operatorname{arccot}(iz)$

$i = \sqrt{-1}$

z is a complex variable of the form $a+bi$, where a and b are real numbers, and i is the imaginary unit

Circular Functions Unit Circle Values

θ (Radians)	θ (Degrees)	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$	$\csc(\theta)$	$\sec(\theta)$	$\cot(\theta)$
0	0°	0	1	0	undefined	1	undefined
$\pi/6$	30°	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$	2	$2\sqrt{3}/3$	$\sqrt{3}$
$\pi/4$	45°	$\sqrt{2}/2$	$\sqrt{2}/2$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\pi/3$	60°	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$	$2\sqrt{3}/3$	2	$\sqrt{3}/3$
$\pi/2$	90°	1	0	undefined	1	undefined	0
$2\pi/3$	120°	$\sqrt{3}/2$	$-1/2$	$-\sqrt{3}$	$2\sqrt{3}/3$	-2	$-\sqrt{3}/3$
$3\pi/4$	135°	$\sqrt{2}/2$	$-\sqrt{2}/2$	-1	$\sqrt{2}$	$-\sqrt{2}$	-1
$5\pi/6$	150°	$1/2$	$-\sqrt{3}/2$	$-\sqrt{3}/3$	2	$-2\sqrt{3}/3$	$-\sqrt{3}$
π	180°	0	-1	0	undefined	-1	undefined
$7\pi/6$	210°	$-1/2$	$-\sqrt{3}/2$	$\sqrt{3}/3$	-2	$-2\sqrt{3}/3$	$\sqrt{3}$
$5\pi/4$	225°	$-\sqrt{2}/2$	$-\sqrt{2}/2$	1	$-\sqrt{2}$	$-\sqrt{2}$	1
$4\pi/3$	240°	$-\sqrt{3}/2$	$-1/2$	$\sqrt{3}$	$-2\sqrt{3}/3$	-2	$\sqrt{3}/3$



Circular Functions Unit Circle Values (cont)

$3\pi/2$	270°	-1	0	undefined	-1	undefined	0
$5\pi/3$	300°	$-\sqrt{3}/2$	$1/2$	$-\sqrt{3}$	$-2\sqrt{3}/3$	2	$-\sqrt{3}/3$
$7\pi/4$	315°	$-\sqrt{2}/2$	$\sqrt{2}/2$	-1	$-\sqrt{2}$	$\sqrt{2}$	-1
$11\pi/6$	330°	-1/2	$\sqrt{3}/2$	$-\sqrt{3}/3$	-2	$2\sqrt{3}/3$	$-\sqrt{3}$
2π	360°	0	1	0	undefined	1	undefined

The coordinates $(\cos(\theta), \sin(\theta))$ represent x and y coordinates of θ on the unit circle $x^2+y^2=1$

Circular Compositional Identities

Composition	$\sin(x)$	$\cos(x)$	$\tan(x)$
$\arcsin(x)$	x	$\sqrt{1-x^2}$	$x/\sqrt{1-x^2}$
$\arccos(x)$	$\sqrt{1-x^2}$	x	$\sqrt{1-x^2}/x$
$\arctan(x)$	$x/\sqrt{1+x^2}$	$1/\sqrt{1+x^2}$	x
$\operatorname{arccsc}(x)$	$1/x$	$\sqrt{x^2-1}/ x $	$\pm 1/\sqrt{x^2-1}$
$\operatorname{arcsec}(x)$	$\sqrt{x^2-1}/ x $	$1/x$	$\pm \sqrt{x^2-1}$
$\operatorname{arccot}(x)$	$1/\sqrt{1+x^2}$	$x/\sqrt{1+x^2}$	$1/x$

Each composition is valid on different domains

Hyperbolic Compositional Identities

Composition	$\sinh(x)$	$\cosh(x)$	$\tanh(x)$
$\operatorname{arcsinh}(x)$	x	$\sqrt{1+x^2}$	$x/\sqrt{1+x^2}$
$\operatorname{arccosh}(x)$	$\sqrt{x^2-1}$	x	$\sqrt{x^2-1}/x$
$\operatorname{arctanh}(x)$	$x/\sqrt{1-x^2}$	$1/\sqrt{1-x^2}$	x
$\operatorname{arccsch}(x)$	$1/x$	$\sqrt{x^2+1}/ x $	$1/\sqrt{x^2+1}$
$\operatorname{arcsech}(x)$	$\sqrt{1-x^2}/x$	$1/x$	$\sqrt{1-x^2}$
$\operatorname{arcoth}(x)$	$x/\sqrt{1-x^2}$	$ x /\sqrt{x^2-1}$	$1/x$

Each composition is valid on different domains

Circular Quotient & Reciprocal Identities

Cofunctional Phase Shift Properties (cont)

Cosecant Complimentary	$\csc(\theta) = \sec(\pi/2 - \theta)$
Cosecant Supplementary	$\csc(\theta) = \csc(\pi - \theta)$
Secant Complimentary	$\sec(\theta) = \csc(\pi/2 - \theta)$
Secant Supplementary	$\sec(\theta) = -\sec(\pi - \theta)$
Cotangent Complimentary	$\cot(\theta) = \tan(\pi/2 - \theta)$
Cotangent Supplementary	$\cot(\theta) = -\cot(\pi - \theta)$

$$n \in \mathbb{N}_1 = \{1, 2, 3, 4, 5, \dots\}$$

Periodicity Properties

Circular Parity Properties

Sine Odd	$\sin(-\theta) = -\sin(\theta)$
Cosine Even	$\cos(-\theta) = \cos(\theta)$
Tangent Odd	$\tan(-\theta) = -\tan(\theta)$
Cosecant Odd	$\csc(-\theta) = -\csc(\theta)$
Secant Even	$\sec(-\theta) = \sec(\theta)$
Cotangent Odd	$\cot(-\theta) = -\cot(\theta)$

Circular Pythagorean Identities

(C) Half/Multiple-Angle Identities (cont)

Tangent Quotient	$\tan(\theta)=\sin(\theta)/\cos(\theta)$	Sine Periodicity	$\sin(\theta)=\sin(\theta-\pm 2\pi n)$	Sine-Cosine Pythagorean	$\sin^2(\theta)+\cos^2(\theta)=1$	Tangent Half-Angle 2	$\tan(\theta/2)=(1-\cos(\theta))/\sin(\theta)$
Cotangent Quotient	$\cot(\theta)=\cos(\theta)/\sin(\theta)$	Cosine Periodicity	$\cos(\theta)=\cos(\theta-\pm 2\pi n)$	Secant-Tangent Pythagorean	$\tan^2(\theta)+1=\sec^2(\theta)$	Tangent Half-Angle 3	$\tan(\theta/2)=\sin(\theta)/(1+\cos(\theta))$
Sine Reciprocal	$\sin(\theta)=1/\csc(\theta)$	Tangent Periodicity	$\tan(\theta)=\tan(\theta\pm\pi n)$	Cosecant-Cotangent Pythagorean	$1+\cot^2(\theta)=\csc^2(\theta)$	Sine Double-Angle 1	$\sin(2\theta)=2\sin(\theta)\cos(\theta)$
Cosine Reciprocal	$\cos(\theta)=1/\sec(\theta)$	Cosecant Periodicity	$\csc(\theta)=\csc(\theta-\pm 2\pi n)$	The last two Pythagorean Identities are obtained by dividing all the terms of the original Sine-Cosine Identity by $\cos^2(\theta)$ and $\sin^2(\theta)$, respectively		Sine Double-Angle 2	$\sin(2\theta)=2\tan(\theta)/(1+\tan^2(\theta))$
Tangent Reciprocal	$\tan(\theta)=1/\cot(\theta)$	Secant Periodicity	$\sec(\theta)=\sec(\theta-\pm 2\pi n)$			Cosine Double-Angle 1	$\cos(2\theta)=\cos^2(\theta)-\sin^2(\theta)$
Cosecant Reciprocal	$\csc(\theta)=1/\sin(\theta)$	Cotangent Periodicity	$\cot(\theta)=\cot(\theta\pm\pi n)$			Cosine Double-Angle 2	$\cos(2\theta)=2\cos^2(\theta)-1$
Secant Reciprocal	$\sec(\theta)=1/\cos(\theta)$	$n \in \mathbb{N}_1 = \{1,2,3,4,5,...\}$		(C) Half/Multiple-Angle Identities			
Cotangent Reciprocal	$\cot(\theta)=1/\tan(\theta)$			Sine Half-Angle	$\sin(\theta/2)=\pm\sqrt{1/2(1-\cos(\theta))}$	Cosine Double-Angle 3	$\cos(2\theta)=1-2\sin^2(\theta)$
All the following identities are true for values that do not cause division by zero				Cosine Half-Angle	$\cos(\theta/2)=\pm\sqrt{1/2(1+\cos(\theta))}$	Cosine Double-Angle 4	$\cos(2\theta)=(1-\tan^2(\theta))/(-1+\tan^2(\theta))$
				Tangent Half-Angle 1	$\tan(\theta/2)=\pm\sqrt{(1-\cos(\theta))/(1+\cos(\theta))}$	Tangent Double-Angle 1	$\tan(2\theta)=2\tan(\theta)/(1-\tan^2(\theta))$
Cofunctional Phase Shift Properties		Sine Complementary	$\sin(\theta)=\cos(\pi/2-\theta)$	Tangent Double-Angle 2	$\tan(2\theta)=2/(\cot(-\theta)-\tan(\theta))$	Sine Triple-Angle	$\sin(3\theta)=3\sin(\theta)-4\sin^3(\theta)$
		Sine Supplementary	$\sin(\theta)=\sin(\pi-\theta)$				
		Cosine Complementary	$\cos(\theta)=\sin(\pi/2-\theta)$				
		Cosine Supplementary	$\cos(\theta)=-\cos(\pi-\theta)$				



(C) Half/Multiple-Angle Identities (cont)

Cosine Triple-Angle	$\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$
Tangent Triple-Angle	$\tan(3\theta) = (3\tan(\theta) - \tan^3(\theta)) / (1 - 3\tan^2(\theta))$

Sine Multiple-Angle Formula:
 $\sin(n\theta) = \sum_{k=0}^n \binom{n}{k} \cos^k(\theta) \sin^{n-k}(\theta) \sin((\pi/2)(n-k))$
 Cosine Multiple-Angle Formula:
 $\cos(n\theta) = \sum_{k=0}^n \binom{n}{k} \cos^k(\theta) \sin^{n-k}(\theta) \cos((\pi/2)(n-k))$
 All the following identities are true for values that do not cause division by zero

Circular Sum/Difference/Product Identities

Sine Sum/Difference	$\sin(\theta \pm \phi)$	$\sin(\theta) \cos(\phi) \pm \cos(\theta) \sin(\phi)$
---------------------	-------------------------	---

Sine Sum-Product	$\sin(\theta) \pm \sin(\phi)$	$2\sin((\theta \pm \phi)/2) \cos((\theta \mp \phi)/2)$
------------------	-------------------------------	--

Sine Product-Sum	$\sin(\theta) \sin(\phi)$	$\frac{1}{2}(\cos(\theta - \phi) - \cos(\theta + \phi))$
------------------	---------------------------	--

Cosine Sum/Difference	$\cos(\theta \pm \phi)$	$\cos(\theta) \cos(\phi) \mp \sin(\theta) \sin(\phi)$
-----------------------	-------------------------	---

Cosine Sum-Product	$\cos(\theta) \pm \cos(\phi)$	$2\cos((\theta \pm \phi)/2) \cos((\theta \mp \phi)/2)$
--------------------	-------------------------------	--

Circular Sum/Difference/Product Identities (cont)

Cosine Product-Sum	$\cos(\theta) \cos(\phi)$	$\frac{1}{2}(\cos(\theta - \phi) + \cos(\theta + \phi))$
--------------------	---------------------------	--

Sine-Cosine Product-Sum	$\sin(\theta) \cos(\phi)$	$\frac{1}{2}(\sin(\theta - \phi) + \sin(\theta + \phi))$
-------------------------	---------------------------	--

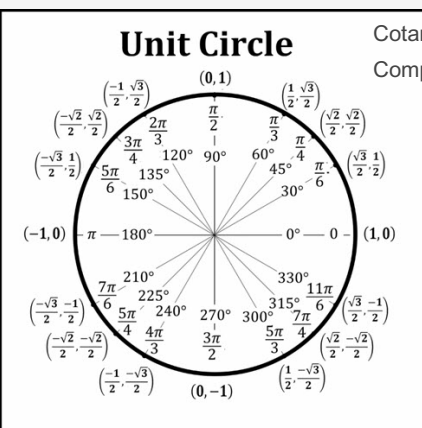
Tangent Sum/Difference	$\tan(\theta \pm \phi)$	$(\tan(\theta) \pm \tan(\phi)) / (1 \mp \tan(\theta) \tan(\phi))$
------------------------	-------------------------	---

Tangent Sum	$\tan(\theta) \pm \tan(\phi)$	$\sin(\theta \pm \phi) / (\cos(\theta) \cos(\phi))$
-------------	-------------------------------	---

Tangent Product	$\tan(\theta) \tan(\phi)$	$(\tan(\theta) + \tan(\phi)) / (\cot(\theta) + \cot(\phi))$
-----------------	---------------------------	---

Tangent-Cotangent Product	$\tan(\theta) \cot(\phi)$	$(\tan(\theta) + \cot(\phi)) / (\cot(\theta) + \tan(\phi))$
---------------------------	---------------------------	---

Sine and Cosine Unit Circle



$$x^2 + y^2 = 1$$

Circular-Inverse Reciprocal Identities

Sine Reciprocal	$\arcsin(1/x) = \arccsc(x)$
Cosine Reciprocal	$\arccos(1/x) = \operatorname{arcsec}(x)$

Tangent Reciprocal 1	$\arctan(1/x) = \operatorname{arccot}(x), x > 0$
Tangent Reciprocal 2	$\arctan(1/x) = \operatorname{arccot}(x) - \pi, x < 0$

Cosecant Reciprocal	$\operatorname{arccsc}(1/x) = \arcsin(x)$
Secant Reciprocal	$\operatorname{arcsec}(1/x) = \arccos(x)$

Cotangent Reciprocal 1	$\operatorname{arccot}(1/x) = \arctan(x), x > 0$
Cotangent Reciprocal 2	$\operatorname{arccot}(1/x) = \arctan(x) + \pi, x < 0$

Circular-Inverse Complimentary Identities

Sine Complimentary	$\arcsin(x) = \pi/2 - \arccos(x)$
--------------------	-----------------------------------

Cosine Complimentary	$\arccos(x) = \pi/2 - \arcsin(x)$
----------------------	-----------------------------------

Tangent Complimentary	$\arctan(x) = \pi/2 - \operatorname{arccot}(x)$
-----------------------	---

Cosecant Complimentary	$\operatorname{arccsc}(x) = \pi/2 - \operatorname{arcsec}(x)$
------------------------	---

Secant Complimentary	$\operatorname{arcsec}(x) = \pi/2 - \operatorname{arccsc}(x)$
----------------------	---

Cotangent Complimentary	$\operatorname{arccot}(x) = \pi/2 - \arctan(x)$
-------------------------	---

Circular-Inverse Negative Input Identities

Sine Odd	$\arcsin(-x) = -\arcsin(x)$
----------	-----------------------------

Cosine Translation	$\arccos(-x) = \pi - \arccos(x)$
--------------------	----------------------------------

Tangent Odd	$\arctan(-x) = -\arctan(x)$
-------------	-----------------------------

Cosecant Odd	$\operatorname{arccsc}(-x) = -\operatorname{arccsc}(x)$
--------------	---

Secant Translation	$\operatorname{arcsec}(-x) = \pi - \operatorname{arcsec}(x)$
--------------------	--

Cotangent Translation	$\operatorname{arccot}(-x) = \pi - \operatorname{arccot}(x)$
-----------------------	--

(CI) Half/Multiple Substitution Identities

Half Sine Substitution 1	$\frac{1}{2}\arcsin(x) = \arcsin(\sqrt{(1+x)/2}) - \pi/4$
--------------------------	---

Half Sine Substitution 2	$\frac{1}{2}\arcsin(x) = \pi/2 - \arcsin(\sqrt{(1-x)/2})$
--------------------------	---

Half Cosine Substitution 1	$\frac{1}{2}\arccos(x) = \arccos(\sqrt{(1+x)/2})$
----------------------------	---

Half Cosine Substitution 2	$\frac{1}{2}\arccos(x) = \pi/2 - \arccos(\sqrt{(1-x)/2})$
----------------------------	---

Double Sine Substitution	$2\arcsin(x) = \arcsin(2x\sqrt{1-x^2}), x \leq \pi/2$
--------------------------	---

Double Cosine Substitution 1	$2\arccos(x) = \arccos(2x^2 - 1), x \geq 0$
------------------------------	---

(CI) Half/Multiple Substitution Identities (cont)

Double Cosine Substitution 2

$$2\arccos(x) = 2\pi - \arccos(2x^2 - 1), \quad x \leq 0$$

Double Tangent Substitution 1

$$2\arctan(x) = \arcsin\left(\frac{2x}{1+x^2}\right), \quad |x| \leq 1$$

Double Tangent Substitution 2

$$2\arctan(x) = \pm \arccos\left(\frac{1-x^2}{1+x^2}\right)$$

Double Tangent Substitution 3

$$2\arctan(x) = \arctan\left(\frac{2x}{1-x^2}\right), \quad |x| < 1$$

Triple Sine Substitution 1

$$3\arcsin(x) = \arcsin(3x - 4x^3), \quad |x| \leq \frac{1}{2}$$

Triple Sine Substitution 2

$$3\arcsin(x) = \arcsin(4x^3 - 3x) \pm \pi, \quad |x| \geq \frac{1}{2}$$

Triple Cosine Substitution 1

$$3\arccos(x) = \arccos(3x - 4x^3), \quad |x| \leq \frac{1}{2}$$

Triple Cosine Substitution 2

$$3\arccos(x) = \arccos(4x^3 - 3x) \pm \pi, \quad |x| \geq \frac{1}{2}$$

Triple Tangent Substitution 1

$$3\arctan(x) = \arctan\left(\frac{3x - x^3}{1 - 3x^2}\right), \quad |x| \leq \sqrt{3}/3$$

Triple Tangent Substitution 2

$$3\arctan(x) = \arctan\left(\frac{3x - x^3}{1 - 3x^2}\right) \pm \pi, \quad |x| \geq \sqrt{3}/3$$

Circular-Inverse Sum/Difference Identities

Sine Sum/Difference

$$\arcsin(x) \pm \arcsin(y) = \arcsin(x\sqrt{1-y^2} \pm y\sqrt{1-x^2})$$

Cosine Sum/Difference

$$\arccos(x) \pm \arccos(y) = \arccos(xy \mp \sqrt{1-x^2}\sqrt{1-y^2})$$

Cosine-Sine Sum/Difference

$$\arccos(x) \pm \arcsin(y) = \arccos(x\sqrt{1-y^2} \mp y\sqrt{1-x^2})$$

Tangent Sum/Difference

$$\arctan(x) \pm \arctan(y) = \arctan\left(\frac{x \pm y}{1 \mp xy}\right), \quad 1 \mp xy \neq 0$$

Law of Sines/Cosines/Tangents

Law of Sines 1

$$\sin(\alpha)/a = \sin(\beta)/b = \sin(\gamma)/c$$

Law of Sines 2

$$a/\sin(\alpha) = b/\sin(\beta) = c/\sin(\gamma)$$

Law of Cosines 1

$$a^2 = b^2 + c^2 - 2bc\cos(\alpha)$$

Law of Cosines 2

$$b^2 = a^2 + c^2 - 2ac\cos(\beta)$$

Law of Cosines 3

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma)$$

Law of Tangents 1

$$(a-b)/(a+b) = \tan((\alpha-\beta)/2)/\tan((\alpha+\beta)/2)$$

Law of Sines/Cosines/Tangents (cont)

Law of Tangents 2

$$(b-c)/(b+c) = \tan((\beta-\gamma)/2)/\tan((\beta+\gamma)/2)$$

Law of Tangents 3

$$(c-a)/(c+a) = \tan((\gamma-\alpha)/2)/\tan((\gamma+\alpha)/2)$$

Side lengths a, b, and c are opposite of the angles α , β , and γ , respectively.

Measurements And Formulas

Radians-Degrees

$$1 \text{ radian} = 180/\pi \text{ degrees; } 1^\circ = (180/\pi)^\circ$$

Degrees-Radians

$$1 \text{ degree} = \pi/180 \text{ radians; } 1^\circ = \pi/180 \text{ radians}$$

Degrees, Minutes, and Seconds (DMS)

$$1 \text{ degree} = 60 \text{ minutes} = 3600 \text{ seconds; } 1^\circ = 60' = 3600''$$

Arc Length/Angular Displacement

$$s = r\theta \text{ units}$$

Sector Area

$$\frac{1}{2}r^2\theta \text{ units}^2$$

Area of a Triangle

$$A_T = \frac{1}{2}bh \text{ units}^2$$

Area of a Circle

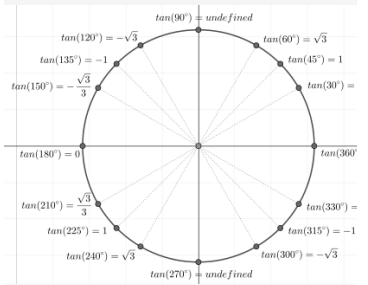
$$A_C = \pi r^2 \text{ units}^2$$

Pythagorean Theorem

$$a^2 + b^2 = c^2$$

Radians are unitless
a, b, and c are side lengths of a right-triangle

Tangent Unit Circle



(H) Quotient & Reciprocal Identities

Tangent Quotient

$$\tanh(\theta) = \sinh(-\theta)/\cosh(\theta)$$

Cotangent Quotient

$$\coth(\theta) = \cosh(-\theta)/\sinh(\theta)$$

Sine Reciprocal

$$\sinh(\theta) = 1/\cosh(\theta)$$

Cosine Reciprocal

$$\cosh(\theta) = 1/\sinh(\theta)$$

Tangent Reciprocal

$$\tanh(\theta) = 1/\coth(\theta)$$

Cosecant Reciprocal

$$\csch(\theta) = 1/\sinh(\theta)$$

Secant Reciprocal

$$\operatorname{sech}(\theta) = 1/\cosh(\theta)$$

Cotangent Reciprocal

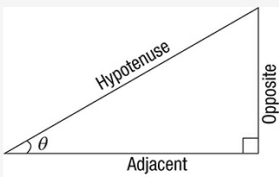
$$\coth(\theta) = 1/\tanh(\theta)$$

All the following identities are true for values that do not cause division by zero

Hyperbolic Parity Properties		(H) Half-Angle & Multiple-Angle Identities (cont)		(H) Half-Angle & Multiple-Angle Identities (cont)		(H) Sum/Difference/Product Identities (cont)			
Sine Odd	$\sinh(-\theta)=-\sinh(\theta)$	Tangent Half-Angle 1	$\tanh(\theta/2)=\pm\sqrt{((\cosh(\theta)-1)/(\cosh(\theta)+1))}$	Sine Triple-Angle	$\sinh(3\theta)=3\sinh(\theta)+4\sinh^3(\theta)$	Cosine Sum-Product 2	$\cosh(\theta)-\cosh(\varphi)$	$2\sinh((- \theta+ \varphi)/2)\sinh((\theta+\varphi)/2)$	
Cosine Even	$\cosh(-\theta)=\cosh(\theta)$	Tangent Half-Angle 2	$\tanh(\theta/2)=(\cosh(\theta)-1)/\sinh(\theta)$	Cosine Triple-Angle	$\cosh(3\theta)=4\cosh^3(\theta)-3\cosh(\theta)$	Cosine Product-Sum	$\cosh(\theta)\cosh(\varphi)$	$\frac{1}{2}(\cosh(\theta+\varphi)+\cosh(-\theta-\varphi))$	
Tangent Odd	$\tanh(-\theta)=-\tanh(\theta)$	Tangent Half-Angle 3	$\tanh(\theta/2)=\sinh(\theta)/(\cosh(\theta)+1)$	Tangent Triple-Angle	$\tanh(3\theta)=(3\tanh(\theta)+\tanh^3(\theta))/(1+3\tanh^2(\theta))$				
Cosecant Odd	$\operatorname{csch}(-\theta)=-\operatorname{csch}(\theta)$	Sine Double-Angle 1	$\sinh(2\theta)=2\sinh(\theta)\cosh(\theta)$	(H) Sum/Difference/Product Identities					Sine-Cosine Product-Sum
Secant Even	$\operatorname{sech}(-\theta)=\operatorname{sech}(\theta)$			Sine Double-Angle 2	$\sinh(2\theta)=2\tanh(\theta)/(1-\tanh^2(\theta))$	Sine Sum-Difference	$\sinh(\theta\pm\varphi)=\sinh(\theta)\cosh(\varphi)\pm\cosh(\theta)\sinh(\varphi)$	Tangent Sum/Difference	$\tanh(\theta\pm\varphi)=(\tanh(\theta)\pm\tanh(\varphi))/(1\pm\tanh(\theta)\tanh(\varphi))$
Cotangent Odd	$\operatorname{coth}(-\theta)=-\operatorname{coth}(\theta)$			Cosine Double-Angle 1	$\cosh(2\theta)=\cosh^2(\theta)+\sinh^2(\theta)$	Sine Sum-Product	$\sinh(\theta)\pm\sinh(\varphi)=2\sinh((- \theta\pm \varphi)/2)\cosh((- \theta\mp \varphi)/2)$	Tangent Sum	$\tanh(\theta)\pm\tanh(\varphi)=\sinh(\theta\pm\varphi)/(\cosh(\theta)\cosh(\varphi))$
Hyperbolic Pythagorean Identities		Cosine Double-Angle 2	$\cosh(2\theta)=2\cosh^2(\theta)-1$	Cosine Sum-Difference	$\cosh(\theta\pm\varphi)=\cosh(\theta)\cosh(\varphi)\pm\sinh(\theta)\sinh(\varphi)$	Tangent Product	$\tanh(\theta)\tanh(\varphi)$	$(\tanh(-\theta)+\tanh(\varphi))/(-\coth(-\theta)+\coth(\varphi))$	
Sine-Cosine Pythagorean	$\cosh^2(\theta)-\sinh^2(\theta)=1$	Cosine Double-Angle 3	$\cosh(2\theta)=-1+2\sinh^2(\theta)$	Sine Product-Sum	$\sinh(\theta)\sinh(\varphi)=\frac{1}{2}(\cosh(\theta+\varphi)-\cosh(\theta-\varphi))$				
Secant Pythagorean	$1-\tanh^2(\theta)=\operatorname{sech}^2(\theta)$	Cosine Double-Angle 4	$\cosh(2\theta)=(-1+\tanh^2(\theta))/(1-\tanh^2(\theta))$	Cosine Sum-Difference	$\cosh(\theta\pm\varphi)=\cosh(-\theta)\cosh(\varphi)\mp\sinh(-\theta)\sinh(\varphi)$				
Cosecant Pythagorean	$\coth^2(\theta)-1=\operatorname{csch}^2(\theta)$	Tangent Double-Angle 1	$\tanh(2\theta)=2\tanh(\theta)/(1+\tanh^2(\theta))$	Cosine Sum-Product 1	$\cosh(\theta)+\cosh(\varphi)=2\cosh((- \theta+ \varphi)/2)\cosh((\theta-- \varphi)/2)$	Tangent-Cotangent Product	$\tanh(\theta)\coth(\varphi)$	$(\tanh(-\theta)+\coth(\varphi))/(-\coth(\theta)+\tanh(\varphi))$	
The last two Hyperbolic Pythagorean Identities are obtained by dividing all the terms of the original Sine-Cosine Identity by $\cosh^2(\theta)$ and $\sinh^2(\theta)$, respectively		Tangent Double-Angle 2	$\tanh(2\theta)=2/(\coth(\theta)+\tanh(\theta))$						
(H) Half-Angle & Multiple-Angle Identities									
Sine Half-Angle	$\sinh(\theta/2)=\pm\sqrt{(\frac{1}{2}(\cosh(\theta)-1))}$								
Cosine Half-Angle	$\cosh(\theta/2)=\sqrt{(\frac{1}{2}(\cosh(\theta)+1))}$								



Right-Triangle Relations



$$\begin{aligned}\sin \theta &= \frac{\text{opposite}}{\text{hypotenuse}} & \csc \theta &= \frac{\text{hypotenuse}}{\text{opposite}} \\ \cos \theta &= \frac{\text{adjacent}}{\text{hypotenuse}} & \sec \theta &= \frac{\text{hypotenuse}}{\text{adjacent}} \\ \tan \theta &= \frac{\text{opposite}}{\text{adjacent}} & \cot \theta &= \frac{\text{adjacent}}{\text{opposite}}\end{aligned}$$

Hyperbolic-Inverse Reciprocal Identities

Sine Reciprocal	$\operatorname{arcsinh}(1/x) = \operatorname{arccsch}(x)$
Cosine Reciprocal	$\operatorname{arccosh}(1/x) = \operatorname{arcsech}(x)$
Tangent Reciprocal	$\operatorname{arctanh}(1/x) = \operatorname{arccoth}(x)$
Cosecant Reciprocal	$\operatorname{arccsch}(1/x) = \operatorname{arcsinh}(x)$
Secant Reciprocal	$\operatorname{arcsech}(1/x) = \operatorname{arccosh}(x)$
Cotangent Reciprocal	$\operatorname{arccoth}(1/x) = \operatorname{arctanh}(x)$

(HI) Negative Input Identities

Inverse Sine Odd	$\operatorname{arcsinh}(-x) = -\operatorname{arcsinh}(x)$
Inverse Tangent Odd	$\operatorname{arctanh}(-x) = -\operatorname{arctanh}(x)$
Inverse Cosecant Odd	$\operatorname{arccsch}(-x) = -\operatorname{arccsch}(x)$
Inverse Cotangent Odd	$\operatorname{arccoth}(-x) = -\operatorname{arccoth}(x)$

(HI) Half/Multiple Substitution Identities

Half Sine Substitution	$\frac{1}{2}\operatorname{arcsinh}(x) = \pm \operatorname{arcsinh}(\sqrt{\frac{1}{2}((1+x^2)-1)/2})$
Half Cosine Substitution	$\frac{1}{2}\operatorname{arccosh}(x) = \operatorname{arccosh}(\sqrt{\frac{1}{2}((x+1)/2)})$
Half Tangent Substitution	$\frac{1}{2}\operatorname{arctanh}(x) = \operatorname{arctanh}(x/(1+\sqrt{1-x^2}))$
Double Sine Substitution 1	$2\operatorname{arcsinh}(x) = \operatorname{arcsinh}(2x\sqrt{1+x^2})$
Double Sine Substitution 2	$2\operatorname{arcsinh}(x) = \pm \operatorname{arccosh}(2x^2+1)$
Double Cosine Substitution	$2\operatorname{arccosh}(x) = \operatorname{arccosh}(2x^2-1), x \geq 1$
Double Tangent Substitution	$2\operatorname{arctanh}(x) = \operatorname{arctanh}(2x/(1+x^2)), x < 1$
Triple Sine Substitution	$3\operatorname{arcsinh}(x) = \operatorname{arcsinh}(3x+4x^3)$
Triple Cosine Substitution	$3\operatorname{arccosh}(x) = \operatorname{arccosh}(4x^3-3x)$
Triple Tangent Substitution	$3\operatorname{arctanh}(x) = \operatorname{arctanh}((3x+x^3)/(1+3x^2))$

(HI) Sum/Difference Identities

Sine Sum/Difference	$\operatorname{arcsinh}(x) \pm \operatorname{arcsinh}(y) = \operatorname{arcsinh}(x\sqrt{y^2+1} \pm y\sqrt{x^2+1})$
Cosine Sum/Difference	$\operatorname{arccosh}(x) \pm \operatorname{arccosh}(y) = \operatorname{arccosh}(xy \pm \sqrt{(x^2-1)(y^2-1)})$
Sine-Cosine Sum/Difference 1	$\operatorname{arcsinh}(x) \pm \operatorname{arccosh}(y) = \operatorname{arcsinh}(xy \pm \sqrt{(x^2+1)(y^2-1)})$
Sine-Cosine Sum/Difference 2	$\operatorname{arcsinh}(x) \pm \operatorname{arccosh}(y) = \pm \operatorname{arccosh}(y\sqrt{x^2+1} \pm x\sqrt{y^2-1})$
Tangent Sum/Difference	$\operatorname{arctanh}(x) \pm \operatorname{arctanh}(y) = \operatorname{arctanh}((x \pm y)/(1 \pm xy))$

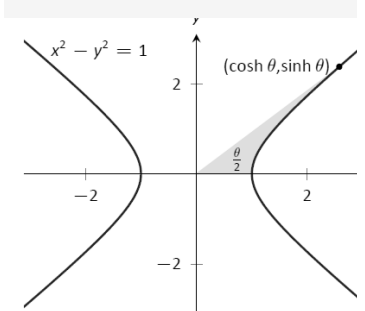
(HI) Logarithmic/Compositional Conversions

Sine Logarithmic	$\ln(x) = \operatorname{arcsinh}((x^2-1)/2x), x > 0$
Cosine Logarithmic	$\ln(x) = \pm \operatorname{arccosh}((x^2+1)/2x)$
Tangent Logarithmic	$\ln(x) = \operatorname{arctanh}((x^2-1)/(x^2+1))$
Sine Hyperbolic-Circular 1	$\operatorname{arcsinh}(\tan(x)) = \ln(\sec(x+\pi n) + \tan(x+\pi n))$
Sine Hyperbolic-Circular 2	$\operatorname{arcsinh}(\tan(x)) = \pm \operatorname{arccosh}(\sec(x+\pi n))$

(HI) Logarithmic/Compositional Conversions (cont)

Tangent Hyperbolic-Circular 1	$\operatorname{arctanh}(\cos(-2x)) = \ln(\cot(x))$
Tangent Hyperbolic-Circular 2	$\pm \operatorname{arctanh}(\sin(x)) = \pm \operatorname{arcsinh}(\tan(x))$
Sine Cofunctional 1	$\operatorname{arcsinh}(x) = \operatorname{arctanh}(x/\sqrt{1+x^2})$
Sine Cofunctional 2	$\operatorname{arcsinh}(x) = \pm \operatorname{arccosh}(\sqrt{1+x^2})$
Cosine Cofunctional 1	$\operatorname{arccosh}(x) = \operatorname{arcsinh}(\sqrt{x^2-1}) , x \geq 1$
Cosine Cofunctional 2	$\operatorname{arccosh}(x) = \operatorname{arctanh}(\sqrt{x^2-1}/x) , x \geq 1$
Tangent Cofunctional 1	$\operatorname{arctanh}(x) = \operatorname{arcsinh}(x/\sqrt{1-x^2})$
Tangent Cofunctional 2	$\operatorname{arctanh}(x) = \pm \operatorname{arccosh}(1/\sqrt{1-x^2})$

Unit Hyperbola



$$x^2 - y^2 = 1$$