

Circular Functions Definitions

Name	Right-Triangle Definition	Domain	Range
Sine Function	$\sin(\theta)=o/h$	$(-\infty, \infty)$	$[-1, 1]$
Cosine Function	$\cos(\theta)=a/h$	$(-\infty, \infty)$	$[-1, 1]$
Tangent Function	$\tan(\theta)=o/a$	$\{\theta \theta \neq \pi/2 \pm \pi n\}$	$(-\infty, \infty)$
Cosecant Function	$\csc(\theta)=h/o$	$\{\theta \theta \neq \pm \pi n\}$	$(-\infty, -1] \cup [1, \infty)$
Secant Function	$\sec(\theta)=h/a$	$\{\theta \theta \neq \pi/2 \pm \pi n\}$	$(-\infty, -1] \cup [1, \infty)$
Cotangent Function	$\cot(\theta)=a/o$	$\{\theta \theta \neq \pm \pi n\}$	$(-\infty, \infty)$
Inverse Sine Function	$\arcsin(o/h)=\theta$	$[-1, 1]$	$[-\pi/2, \pi/2]$
Inverse Cosine Function	$\arccos(a/h)=\theta$	$[-1, 1]$	$[0, \pi]$
Inverse Tangent Function	$\arctan(o/a)=\theta$	$(-\infty, \infty)$	$(-\pi/2, \pi/2)$
Inverse Cosecant Function	$\text{arccsc}(h/o)=\theta$	$(-\infty, -1) \cup (1, \infty)$	$[-\pi/2, 0) \cup (0, \pi/2]$
Inverse Secant Function	$\text{arcsec}(h/a)=\theta$	$(-\infty, -1) \cup (1, \infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$
Inverse Cotangent Function	$\text{arccot}(a/o)=\theta$	$(-\infty, \infty)$	$(0, \pi)$
Circular Euler Relation	$e^{\pm i\theta}=\cos(\theta) \pm i\sin(\theta)$		
De Moivre's Theorem	$e^{in\theta}=(\cos(\theta)+i\sin(\theta))^n=\cos(n\theta)+i\sin(n\theta)$		

$n \in \mathbb{N}_1 = \{1, 2, 3, 4, 5, \dots\}$

"h" is the "hypotenuse" leg of a right triangle. It is directly across the right (90°) angle, and it has the longest length of the three sides.

"o" is the "opposite" leg of a right triangle. It is directly across the chosen angle θ .

"a" is the "adjacent" leg of a right triangle. It is the leg that is neither the hypotenuse leg, nor the opposite leg.

By the Pythagorean theorem, $o^2+a^2=h^2$

Hyperbolic Functions Definitions

Name	Exponential Definition	Domain	Range
Hyperbolic Sine Function	$\sinh(\theta)=(e^\theta-e^{-\theta})/2$	$(-\infty, \infty)$	$(-\infty, \infty)$
Hyperbolic Cosine Function	$\cosh(\theta)=(e^\theta+e^{-\theta})/2$	$(-\infty, \infty)$	$[1, \infty)$
Hyperbolic Tangent Function	$\tanh(\theta)=(e^\theta-e^{-\theta})/(e^\theta+e^{-\theta})$	$(-\infty, \infty)$	$(-1, 1)$
Hyperbolic Cosecant Function	$\text{csch}(\theta)=2/(e^\theta-e^{-\theta})$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
Hyperbolic Secant Function	$\text{sech}(\theta)=2/(e^\theta+e^{-\theta})$	$(-\infty, \infty)$	$(0, 1]$



Hyperbolic Functions Definitions (cont)

Hyperbolic Cotangent Function	$\coth(\theta) = (e^\theta + e^{-\theta}) / (e^\theta - e^{-\theta})$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, -1) \cup (1, \infty)$
Inverse Hyperbolic Sine Function	$\operatorname{arcsinh}(x) = \ln(x + \sqrt{x^2 + 1})$	$(-\infty, \infty)$	$(-\infty, \infty)$
Inverse Hyperbolic Cosine Function	$\operatorname{arccosh}(x) = \ln(x + \sqrt{x^2 - 1})$	$[1, \infty)$	$[0, \infty)$
Inverse Hyperbolic Tangent Function	$\operatorname{arctanh}(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$	$(-1, 1)$	$(-\infty, \infty)$
Inverse Hyperbolic Cosecant Function	$\operatorname{arccsch}(x) = \ln\left(\frac{1 \pm \sqrt{1+x^2}}{x}\right)$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
Inverse Hyperbolic Secant Function	$\operatorname{arcsech}(x) = \ln\left(\frac{1 + \sqrt{1-x^2}}{\theta}\right)$	$(0, 1]$	$[0, \infty)$
Inverse Hyperbolic Cotangent Function	$\operatorname{arcoth}(x) = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right)$	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, 0) \cup (0, \infty)$
Hyperbolic Euler Relation	$e^{\pm\theta} = \cosh(\theta) \pm \sinh(\theta)$		
De Moivre's Theorem (Hyperbolic)	$e^{n\theta} = (\cosh(\theta) + \sinh(\theta))^n = \cosh(n\theta) + \sinh(n\theta)$		

$n \in \mathbb{N}_1 = \{1, 2, 3, 4, 5, \dots\}$

Complex Definitions

Name	Complex Relation	Circular-Hyperbolic Relation
Complex Sine	$\sin(z) = (e^{iz} - e^{-iz}) / 2i$	$\sin(z) = -i \sinh(iz)$
Complex Cosine	$\cos(z) = (e^{iz} + e^{-iz}) / 2$	$\cos(z) = \cosh(iz)$
Complex Tangent	$\tan(z) = i(e^{iz} - e^{-iz}) / (e^{iz} + e^{-iz})$	$\tan(z) = i \tanh(iz)$
Complex Cosecant	$\csc(z) = 2i / (e^{iz} - e^{-iz})$	$\csc(z) = i \operatorname{csch}(iz)$
Complex Secant	$\sec(z) = 2 / (e^{iz} + e^{-iz})$	$\sec(z) = \operatorname{sech}(iz)$
Complex Cotangent	$\cot(z) = i(e^{iz} + e^{-iz}) / (e^{iz} - e^{-iz})$	$\cot(z) = i \operatorname{coth}(iz)$
Complex Inverse Sine	$\operatorname{arcsin}(z) = -i \ln(iz \pm \sqrt{1-z^2})$	$\operatorname{arcsin}(z) = -i \operatorname{arcsinh}(iz)$
Complex Inverse Cosine	$\operatorname{arccos}(z) = -i \ln(z \pm \sqrt{1-z^2})$	$\operatorname{arccos}(z) = \pm i \operatorname{arccosh}(z)$
Complex Inverse Tangent	$\operatorname{arctan}(z) = (i/2) \ln\left(\frac{i+z}{i-z}\right)$	$\operatorname{arctan}(z) = -i \operatorname{arctanh}(iz)$
Complex Inverse Cosecant	$\operatorname{arccsc}(z) = -i \ln\left(\frac{i + \sqrt{z^2 - 1}}{z}\right)$	$\operatorname{arccsc}(z) = i \operatorname{arccsch}(iz)$
Complex Inverse Secant	$\operatorname{arcsec}(z) = -i \ln\left(\frac{1 + \sqrt{1-z^2}}{z}\right)$	$\operatorname{arcsec}(z) = \pm i \operatorname{arcsech}(z)$
Complex Inverse Cotangent	$\operatorname{arccot}(z) = -(i/2) \ln\left(\frac{z+i}{z-i}\right)$	$\operatorname{arccot}(z) = \pm i \operatorname{arcoth}(iz)$
Complex Hyperbolic Sine	None	$\sinh(z) = -i \sin(iz)$



Complex Definitions (cont)

Complex Hyperbolic Cosine	None	$\cosh(z)=\cos(iz)$
Complex Hyperbolic Tangent	None	$\tanh(z)=-i\tan(iz)$
Complex Hyperbolic Cosecant	None	$\operatorname{csch}(z)=i\operatorname{csc}(iz)$
Complex Hyperbolic Secant	None	$\operatorname{sech}(z)=\sec(iz)$
Complex Hyperbolic Cotangent	None	$\operatorname{coth}(z)=i\cot(iz)$
Complex Inverse Hyperbolic Sine	None	$\operatorname{arcsinh}(z)=-i\operatorname{arcsin}(iz)$
Complex Inverse Hyperbolic Cosine	None	$\operatorname{arccosh}(z)=\pm i\operatorname{arccos}(z)$
Complex Inverse Hyperbolic Tangent	None	$\operatorname{arctanh}(z)=-i\operatorname{arctan}(iz)$
Complex Inverse Hyperbolic Cosecant	None	$\operatorname{arccsch}(z)=-i\operatorname{arccsc}(z)$
Complex Inverse Hyperbolic Secant	None	$\operatorname{arcsech}(z)=\pm i\operatorname{arcsec}(z)$
Complex Inverse Hyperbolic Cotangent	None	$\operatorname{arcoth}(z)=-i\operatorname{arccot}(iz)$

$$i=\sqrt{-1}$$

z is a complex variable of the form $a+bi$, where a and b are real numbers, and i is the imaginary unit

Circular Functions Unit Circle Values

θ (Radians)	θ (Degrees)	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$	$\operatorname{csc}(\theta)$	$\operatorname{sec}(\theta)$	$\operatorname{cot}(\theta)$
0	0°	0	1	0	undefined	1	undefined
$\pi/6$	30°	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$	2	$2\sqrt{3}/3$	$\sqrt{3}$
$\pi/4$	45°	$\sqrt{2}/2$	$\sqrt{2}/2$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\pi/3$	60°	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$	$2\sqrt{3}/3$	2	$\sqrt{3}/3$
$\pi/2$	90°	1	0	undefined	1	undefined	0
$2\pi/3$	120°	$\sqrt{3}/2$	$-1/2$	$-\sqrt{3}$	$2\sqrt{3}/3$	-2	$-\sqrt{3}/3$
$3\pi/4$	135°	$\sqrt{2}/2$	$-\sqrt{2}/2$	-1	$\sqrt{2}$	$-\sqrt{2}$	-1
$5\pi/6$	150°	$1/2$	$-\sqrt{3}/2$	$-\sqrt{3}/3$	2	$-2\sqrt{3}/3$	$-\sqrt{3}$
π	180°	0	-1	0	undefined	-1	undefined
$7\pi/6$	210°	$-1/2$	$-\sqrt{3}/2$	$\sqrt{3}/3$	-2	$-2\sqrt{3}/3$	$\sqrt{3}$
$5\pi/4$	225°	$-\sqrt{2}/2$	$-\sqrt{2}/2$	1	$-\sqrt{2}$	$-\sqrt{2}$	1
$4\pi/3$	240°	$-\sqrt{3}/2$	$-1/2$	$\sqrt{3}$	$-2\sqrt{3}/3$	-2	$\sqrt{3}/3$



Circular Functions Unit Circle Values (cont)

$3\pi/2$	270°	-1	0	undefined	-1	undefined	0
$5\pi/3$	300°	$-\sqrt{3}/2$	$1/2$	$-\sqrt{3}$	$-2\sqrt{3}/3$	2	$-\sqrt{3}/3$
$7\pi/4$	315°	$-\sqrt{2}/2$	$\sqrt{2}/2$	-1	$-\sqrt{2}$	$\sqrt{2}$	-1
$11\pi/6$	330°	$-1/2$	$\sqrt{3}/2$	$-\sqrt{3}/3$	-2	$2\sqrt{3}/3$	$-\sqrt{3}$
2π	360°	0	1	0	undefined	1	undefined

The coordinates $(\cos(\theta), \sin(\theta))$ represent x and y coordinates of θ on the unit circle $x^2+y^2=1$

Circular Compositional Identities

Composition	$\sin(x)$	$\cos(x)$	$\tan(x)$
$\arcsin(x)$	x	$\sqrt{1-x^2}$	$x/\sqrt{1-x^2}$
$\arccos(x)$	$\sqrt{1-x^2}$	x	$\sqrt{1-x^2}/x$
$\arctan(x)$	$x/\sqrt{1+x^2}$	$1/\sqrt{1+x^2}$	x
$\text{arccsc}(x)$	$1/x$	$\sqrt{x^2-1}/ x $	$\pm 1/\sqrt{x^2-1}$
$\text{arcsec}(x)$	$\sqrt{x^2-1}/ x $	$1/x$	$\pm \sqrt{x^2-1}$
$\text{arccot}(x)$	$1/\sqrt{1+x^2}$	$x/\sqrt{1+x^2}$	$1/x$

Each composition is valid on different domains

Hyperbolic Compositional Identities

Composition	$\sinh(x)$	$\cosh(x)$	$\tanh(x)$
$\text{arcsinh}(x)$	x	$\sqrt{1+x^2}$	$x/\sqrt{1+x^2}$
$\text{arccosh}(x)$	$\sqrt{x^2-1}$	x	$\sqrt{x^2-1}/x$
$\text{arctanh}(x)$	$x/\sqrt{1-x^2}$	$1/\sqrt{1-x^2}$	x
$\text{arcsch}(x)$	$1/x$	$\sqrt{x^2+1}/ x $	$1/\sqrt{x^2+1}$
$\text{arcsech}(x)$	$\sqrt{1-x^2}/x$	$1/x$	$\sqrt{1-x^2}$
$\text{arcoth}(x)$	$x/\sqrt{1-x^2}$	$ x /\sqrt{x^2-1}$	$1/x$

Each composition is valid on different domains

Circular Quotient & Reciprocal Identities

Cofunctional Phase Shift Properties (cont)

Cosecant Complimentary	$\csc(\theta) = \sec(\pi/2 - \theta)$
Cosecant Supplementary	$\csc(\theta) = \csc(\pi - \theta)$
Secant Complimentary	$\sec(\theta) = \csc(\pi/2 - \theta)$
Secant Supplementary	$\sec(\theta) = -\sec(\pi - \theta)$
Cotangent Complimentary	$\cot(\theta) = \tan(\pi/2 - \theta)$
Cotangent Supplementary	$\cot(\theta) = -\cot(\pi - \theta)$

$$n \in \mathbb{N}_1 = \{1, 2, 3, 4, 5, \dots\}$$

Periodicity Properties

Circular Parity Properties

Sine Odd	$\sin(-\theta) = -\sin(\theta)$
Cosine Even	$\cos(-\theta) = \cos(\theta)$
Tangent Odd	$\tan(-\theta) = -\tan(\theta)$
Cosecant Odd	$\csc(-\theta) = -\csc(\theta)$
Secant Even	$\sec(-\theta) = \sec(\theta)$
Cotangent Odd	$\cot(-\theta) = -\cot(\theta)$

Circular Pythagorean Identities

(C) Half/Multiple-Angle Identities (cont)

Tangent Quotient	$\tan(\theta) = \sin(\theta)/\cos(\theta)$
Cotangent Quotient	$\cot(\theta) = \cos(\theta)/\sin(\theta)$
Sine Reciprocal	$\sin(\theta) = 1/\csc(\theta)$
Cosine Reciprocal	$\cos(\theta) = 1/\sec(\theta)$
Tangent Reciprocal	$\tan(\theta) = 1/\cot(\theta)$
Cosecant Reciprocal	$\csc(\theta) = 1/\sin(\theta)$
Secant Reciprocal	$\sec(\theta) = 1/\cos(\theta)$
Cotangent Reciprocal	$\cot(\theta) = 1/\tan(\theta)$

All the following identities are true for values that do not cause division by zero

Cofunctional Phase Shift Properties

Sine Complementary	$\sin(\theta) = \cos(\pi/2 - \theta)$
Sine Supplementary	$\sin(\theta) = \sin(\pi - \theta)$
Cosine Complementary	$\cos(\theta) = \sin(\pi/2 - \theta)$
Cosine Supplementary	$\cos(\theta) = -\cos(\pi - \theta)$
Tangent Complementary	$\tan(\theta) = \cot(\pi/2 - \theta)$
Tangent Supplementary	$\tan(\theta) = -\tan(\pi - \theta)$

Sine Periodicity	$\sin(\theta) = \sin(\theta \pm 2\pi n)$
Cosine Periodicity	$\cos(\theta) = \cos(\theta \pm 2\pi n)$
Tangent Periodicity	$\tan(\theta) = \tan(\theta \pm \pi n)$
Cosecant Periodicity	$\csc(\theta) = \csc(\theta \pm 2\pi n)$
Secant Periodicity	$\sec(\theta) = \sec(\theta \pm 2\pi n)$
Cotangent Periodicity	$\cot(\theta) = \cot(\theta \pm \pi n)$
$n \in \mathbb{N}_1 = \{1, 2, 3, 4, 5, \dots\}$	

Sine-Cosine Pythagorean	$\sin^2(\theta) + \cos^2(\theta) = 1$
Secant-Tangent Pythagorean	$\tan^2(\theta) + 1 = \sec^2(\theta)$
Cosecant-Cotangent Pythagorean	$1 + \cot^2(\theta) = \csc^2(\theta)$
The last two Pythagorean Identities are obtained by dividing all the terms of the original Sine-Cosine Identity by $\cos^2(\theta)$ and $\sin^2(\theta)$, respectively	

(C) Half/Multiple-Angle Identities

Sine Half-Angle	$\sin(\theta/2) = \pm\sqrt{\frac{1}{2}(1 - \cos(\theta))}$
Cosine Half-Angle	$\cos(\theta/2) = \pm\sqrt{\frac{1}{2}(1 + \cos(\theta))}$
Tangent Half-Angle 1	$\tan(\theta/2) = \pm\sqrt{\frac{1 - \cos(\theta)}{1 + \cos(\theta)}}$

Tangent Half-Angle 2	$\tan(\theta/2) = (1 - \cos(\theta))/\sin(\theta)$
Tangent Half-Angle 3	$\tan(\theta/2) = \sin(\theta)/(1 + \cos(\theta))$
Sine Double-Angle 1	$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
Sine Double-Angle 2	$\sin(2\theta) = 2\tan(\theta)/(1 + \tan^2(\theta))$
Cosine Double-Angle 1	$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
Cosine Double-Angle 2	$\cos(2\theta) = 2\cos^2(\theta) - 1$
Cosine Double-Angle 3	$\cos(2\theta) = 1 - 2\sin^2(\theta)$
Cosine Double-Angle 4	$\cos(2\theta) = (1 - \tan^2(\theta))/(1 + \tan^2(\theta))$
Tangent Double-Angle 1	$\tan(2\theta) = 2\tan(\theta)/(1 - \tan^2(\theta))$
Tangent Double-Angle 2	$\tan(2\theta) = 2/(\cot(\theta) - \tan(\theta))$
Sine Triple-Angle	$\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta)$



(C) Half/Multiple-Angle Identities (cont)

Cosine Triple-Angle	$\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$
Tangent Triple-Angle	$\tan(3\theta) = (3\tan(\theta) - \tan^3(\theta)) / (1 - 3\tan^2(\theta))$

Sine Multiple-Angle Formula:
 $\sin(n\theta) = \sum_{k=0}^{n-1} \binom{n}{k} \cos^k(\theta) \sin^{n-k}(\theta) \sin((\pi/2)(n-k))$

Cosine Multiple-Angle Formula:
 $\cos(n\theta) = \sum_{k=0}^{n-1} \binom{n}{k} \cos^k(\theta) \sin^{n-k}(\theta) \cos((\pi/2)(n-k))$

All the following identities are true for values that do not cause division by zero

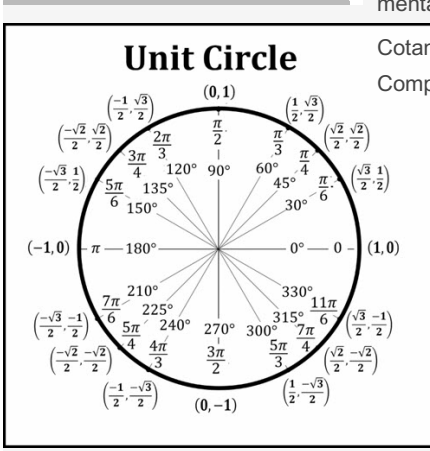
Circular Sum/Difference/Product Identities

Sine Sum/Difference	$\sin(\theta \pm \phi)$	$\sin(\theta) \cos(\phi) \pm \cos(\theta) \sin(\phi)$
Sine Sum-Product	$\sin(\theta) \pm \sin(\phi)$	$2\sin((\theta \pm \phi)/2) \cos((\theta \mp \phi)/2)$
Sine Product-Sum	$\sin(\theta) \sin(\phi)$	$\frac{1}{2}(\cos(\theta - \phi) - \cos(\theta + \phi))$
Cosine Sum/Difference	$\cos(\theta \pm \phi)$	$\cos(\theta) \cos(\phi) \mp \sin(\theta) \sin(\phi)$
Cosine Sum-Product	$\cos(\theta) \pm \cos(\phi)$	$2\cos((\theta \pm \phi)/2) \cos((\theta \mp \phi)/2)$

Circular Sum/Difference/Product Identities (cont)

Cosine Product-Sum	$\cos(\theta) \cos(\phi)$	$\frac{1}{2}(\cos(\theta - \phi) + \cos(\theta + \phi))$
Sine-Cosine Product-Sum	$\sin(\theta) \cos(\phi)$	$\frac{1}{2}(\sin(\theta - \phi) + \sin(\theta + \phi))$
Tangent Sum/Difference	$\tan(\theta \pm \phi)$	$(\tan(\theta) \pm \tan(\phi)) / (1 \mp \tan(\theta)\tan(\phi))$
Tangent Sum	$\tan(\theta) \pm \tan(\phi)$	$\frac{\sin(\theta \pm \phi)}{\cos(\theta) \cos(\phi)}$
Tangent Product	$\tan(\theta) \tan(\phi)$	$(\tan(\theta) + \tan(\phi)) / (\cot(\theta) + \cot(\phi))$
Tangent-Cotangent Product	$\tan(\theta) \cot(\phi)$	$(\tan(\theta) + \cot(\phi)) / (\cot(\theta) + \tan(\phi))$

Sine and Cosine Unit Circle



$$x^2 + y^2 = 1$$

Circular-Inverse Reciprocal Identities

Sine Reciprocal	$\arcsin(1/x) = \arccsc(x)$
Cosine Reciprocal	$\arccos(1/x) = \operatorname{arcsec}(x)$
Tangent Reciprocal 1	$\arctan(1/x) = \operatorname{arccot}(x), x > 0$
Tangent Reciprocal 2	$\arctan(1/x) = \operatorname{arccot}(x) - \pi, x < 0$
Cosecant Reciprocal	$\operatorname{arccsc}(1/x) = \arcsin(x)$
Secant Reciprocal	$\operatorname{arcsec}(1/x) = \arccos(x)$
Cotangent Reciprocal 1	$\operatorname{arccot}(1/x) = \arctan(x), x > 0$
Cotangent Reciprocal 2	$\operatorname{arccot}(1/x) = \arctan(x) + \pi, x < 0$

Circular-Inverse Complimentary Identities

Sine Complimentary	$\arcsin(x) = \pi/2 - \arccos(x)$
Cosine Complimentary	$\arccos(x) = \pi/2 - \arcsin(x)$
Tangent Complimentary	$\arctan(x) = \pi/2 - \operatorname{arccot}(x)$
Cosecant Complimentary	$\operatorname{arccsc}(x) = \pi/2 - \operatorname{arcsec}(x)$
Secant Complimentary	$\operatorname{arcsec}(x) = \pi/2 - \operatorname{arccsc}(x)$
Cotangent Complimentary	$\operatorname{arccot}(x) = \pi/2 - \arctan(x)$

Circular-Inverse Negative Input Identities

Sine Odd	$\arcsin(-x) = -\arcsin(x)$
Cosine Translation	$\arccos(-x) = \pi - \arccos(x)$
Tangent Odd	$\arctan(-x) = -\arctan(x)$
Cosecant Odd	$\operatorname{arccsc}(-x) = -\operatorname{arccsc}(x)$
Secant Translation	$\operatorname{arcsec}(-x) = \pi - \operatorname{arcsec}(x)$
Cotangent Translation	$\operatorname{arccot}(-x) = \pi - \operatorname{arccot}(x)$

(CI) Half/Multiple Substitution Identities

Half Sine Substitution 1	$\frac{1}{2}\arcsin(x) = \arcsin(\sqrt{(1+x)/2}) - \pi/4$
Half Sine Substitution 2	$\frac{1}{2}\arcsin(x) = \arcsin(\sqrt{(1-x)/2})$
Half Cosine Substitution 1	$\frac{1}{2}\arccos(x) = \arccos(\sqrt{(1+x)/2})$
Half Cosine Substitution 2	$\frac{1}{2}\arccos(x) = \arccos(\sqrt{(1-x)/2})$
Double Sine Substitution	$2\arcsin(x) = \arcsin(2x\sqrt{1-x^2}), x \leq \pi/2$
Double Cosine Substitution 1	$2\arccos(x) = \arccos(2x^2 - 1), x \geq 0$



(CI) Half/Multiple Substitution Identities (cont)

Double Cosine Substitution 2
 $2\arccos(x) = 2\pi - \arccos(2x^2 - 1), x \leq 0$

Double Tangent Substitution 1
 $2\arctan(x) = \arcsin(2x/(1+x^2)), |x| \leq 1$

Double Tangent Substitution 2
 $2\arctan(x) = \pm \arccos((1-x^2)/(1+x^2))$

Double Tangent Substitution 3
 $2\arctan(x) = \arctan(2x/(1-x^2)), |x| < 1$

Triple Sine Substitution 1
 $3\arcsin(x) = \arcsin(3x - 4x^3), |x| \leq 1/2$

Triple Sine Substitution 2
 $3\arcsin(x) = \arcsin(4x^3 - 3x) \pm \pi, |x| \geq 1/2$

Triple Cosine Substitution 1
 $3\arccos(x) = \arccos(3x - 4x^3), |x| \leq 1/2$

Triple Cosine Substitution 2
 $3\arccos(x) = \arccos(4x^3 - 3x) + \pi \pm \pi, |x| \geq 1/2$

Triple Tangent Substitution 1
 $3\arctan(x) = \arctan((3x - x^3)/(1 - 3x^2)), |x| \leq \sqrt{3}/3$

Triple Tangent Substitution 2
 $3\arctan(x) = \arctan((3x - x^3)/(1 - 3x^2)) \pm \pi, |x| \geq \sqrt{3}/3$

Circular-Inverse Sum/Difference Identities

Sine Sum/Difference
 $\arcsin(x) \pm \arcsin(y) = \arcsin(x\sqrt{1-y^2} \pm y\sqrt{1-x^2})$

Cosine Sum/Difference
 $\arccos(x) \pm \arccos(y) = \arccos(xy \mp \sqrt{(1-x^2)\sqrt{1-y^2}})$

Cosine-Sine Sum/Difference
 $\arccos(x) \pm \arcsin(y) = \arccos(x\sqrt{1-y^2} \mp y\sqrt{1-x^2})$

Tangent Sum/Difference
 $\arctan(x) \pm \arctan(y) = \arctan((x \pm y)/(1 \mp xy)), 1 \mp xy \neq 0$

Law of Sines/Cosines/Tangents

Law of Sines 1
 $\sin(\alpha)/a = \sin(\beta)/b = \sin(\gamma)/c$

Law of Sines 2
 $a/\sin(\alpha) = b/\sin(\beta) = c/\sin(\gamma)$

Law of Cosines 1
 $a^2 = b^2 + c^2 - 2bc\cos(\alpha)$

Law of Cosines 2
 $b^2 = a^2 + c^2 - 2ac\cos(\beta)$

Law of Cosines 3
 $c^2 = a^2 + b^2 - 2ab\cos(\gamma)$

Law of Tangents 1
 $(a-b)/(a+b) = \tan((\alpha-\beta)/2)/\tan((\alpha+\beta)/2)$

Law of Sines/Cosines/Tangents (cont)

Law of Tangents 2
 $(b-c)/(b+c) = \tan((\beta-\gamma)/2)/\tan((\beta+\gamma)/2)$

Law of Tangents 3
 $(c-a)/(c+a) = \tan((\gamma-\alpha)/2)/\tan((\gamma+\alpha)/2)$

Side lengths a, b, and c are opposite of the angles α , β , and γ , respectively.

Measurements And Formulas

Radians-Degrees
 1 radian = $180/\pi$ degrees; $1 = (180/\pi)^\circ$

Degrees-Radians
 1 degree = $\pi/180$ radians; $1^\circ = \pi/180$ radians

Degrees, Minutes, and Seconds (DMS)
 1 degree = 60 minutes = 3600 seconds; $1^\circ = 60' = 3600''$

Arc Length/Angular Displacement
 $s = r\theta$ units

Sector Area
 $\frac{1}{2}r^2\theta$ units²

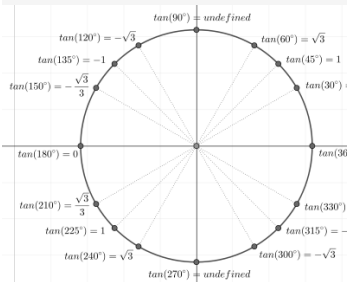
Area of a Triangle
 $A_T = \frac{1}{2}bh$ units²

Area of a Circle
 $A_C = \pi r^2$ units²

Pythagorean Theorem
 $a^2 + b^2 = c^2$

Radians are unitless
 a, b, and c are side lengths of a right-triangle

Tangent Unit Circle



(H) Quotient & Reciprocal Identities

Tangent Quotient
 $\tanh(\theta) = \sinh(\theta)/\cosh(\theta)$

Cotangent Quotient
 $\coth(\theta) = \cosh(\theta)/\sinh(\theta)$

Sine Reciprocal
 $\sinh(\theta) = 1/\operatorname{csch}(\theta)$

Cosine Reciprocal
 $\cosh(\theta) = 1/\operatorname{sech}(\theta)$

Tangent Reciprocal
 $\tanh(\theta) = 1/\operatorname{coth}(\theta)$

Cosecant Reciprocal
 $\operatorname{csch}(\theta) = 1/\sinh(\theta)$

Secant Reciprocal
 $\operatorname{sech}(\theta) = 1/\cosh(\theta)$

Cotangent Reciprocal
 $\operatorname{coth}(\theta) = 1/\tanh(\theta)$

All the following identities are true for values that do not cause division by zero

Hyperbolic Parity Properties

Sine Odd	$\sinh(-\theta) = -\sinh(\theta)$
Cosine Even	$\cosh(-\theta) = \cosh(\theta)$
Tangent Odd	$\tanh(-\theta) = -\tanh(\theta)$
Cosecant Odd	$\operatorname{csch}(-\theta) = -\operatorname{csch}(\theta)$
Secant Even	$\operatorname{sech}(-\theta) = \operatorname{sech}(\theta)$
Cotangent Odd	$\operatorname{coth}(-\theta) = -\operatorname{coth}(\theta)$

Hyperbolic Pythagorean Identities

Sine-Cosine Pythagorean	$\cosh^2(\theta) - \sinh^2(\theta) = 1$
Secant Pythagorean	$1 - \tanh^2(\theta) = \operatorname{sech}^2(\theta)$
Cosecant Pythagorean	$\operatorname{coth}^2(\theta) - 1 = \operatorname{csch}^2(\theta)$

The last two Hyperbolic Pythagorean Identities are obtained by dividing all the terms of the original Sine-Cosine Identity by $\cosh^2(\theta)$ and $\sinh^2(\theta)$, respectively

(H) Half-Angle & Multiple-Angle Identities

Sine Half-Angle	$\sinh(\theta/2) = \pm \sqrt{\frac{1}{2}(\cosh(\theta) - 1)}$
Cosine Half-Angle	$\cosh(\theta/2) = \sqrt{\frac{1}{2}(\cosh(\theta) + 1)}$

(H) Half-Angle & Multiple-Angle Identities (cont)

Tangent Half-Angle 1	$\tanh(\theta/2) = \pm \sqrt{\frac{\cosh(\theta) - 1}{\cosh(\theta) + 1}}$
Tangent Half-Angle 2	$\tanh(\theta/2) = \frac{\cosh(\theta) - 1}{\sinh(\theta)}$
Tangent Half-Angle 3	$\tanh(\theta/2) = \frac{\sinh(\theta)}{\cosh(\theta) + 1}$

Sine Double-Angle 1	$\sinh(2\theta) = 2\sinh(\theta)\cosh(\theta)$
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Sine Double-Angle 2	$\sinh(2\theta) = 2\tanh(\theta)/(1 - \tanh^2(\theta))$
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Cosine Double-Angle 1	$\cosh(2\theta) = \cosh^2(\theta) + \sinh^2(\theta)$
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Cosine Double-Angle 2	$\cosh(2\theta) = 2\cosh^2(\theta) - 1$
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Cosine Double-Angle 3	$\cosh(2\theta) = \frac{1 + 2\sinh^2(\theta)}{1 + \tanh^2(\theta)}$
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Cosine Double-Angle 4	$\cosh(2\theta) = \frac{1 + \tanh^2(\theta)}{1 - \tanh^2(\theta)}$
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Tangent Double-Angle 1	$\tanh(2\theta) = 2\tanh(\theta)/(1 + \tanh^2(\theta))$
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Tangent Double-Angle 2	$\tanh(2\theta) = 2/(\operatorname{coth}(\theta) + \tanh(\theta))$
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(H) Half-Angle & Multiple-Angle Identities (cont)

Sine Triple-Angle	$\sinh(3\theta) = 3\sinh(\theta) + 4\sinh^3(\theta)$
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Cosine Triple-Angle	$\cosh(3\theta) = 4\cosh^3(\theta) - 3\cosh(\theta)$
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Tangent Triple-Angle	$\tanh(3\theta) = \frac{3\tanh(\theta) + \tanh^3(\theta)}{1 + 3\tanh^2(\theta)}$
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(H) Sum/Difference/Product Identities

Sine Sum/Difference	$\sinh(\theta \pm \phi) = \sinh(\theta)\cosh(\phi) \pm \cosh(\theta)\sinh(\phi)$
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Sine Sum-Product	$\sinh(\theta) \pm \sinh(\phi) = 2\sinh\left(\frac{\theta + \phi}{2}\right)\cosh\left(\frac{\theta - \phi}{2}\right)$
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Sine Product-Sum	$\sinh(\theta)\sinh(\phi) = \frac{1}{2}(\cosh(\theta + \phi) - \cosh(\theta - \phi))$
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Cosine Sum/Difference	$\cosh(\theta \pm \phi) = \cosh(\theta)\cosh(\phi) \mp \sinh(\theta)\sinh(\phi)$
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Cosine Sum-Product 1	$\cosh(\theta) \pm \cosh(\phi) = 2\cosh\left(\frac{\theta + \phi}{2}\right)\cosh\left(\frac{\theta - \phi}{2}\right)$
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(H) Sum/Difference/Product Identities (cont)

Cosine Sum-Product 2	$\cosh(\theta)\cosh(\phi) = \frac{1}{2}(\cosh(\theta + \phi) + \cosh(\theta - \phi))$
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Cosine Product-Sum	$\cosh(\theta)\cosh(\phi) = \frac{1}{2}(\cosh(\theta + \phi) + \cosh(\theta - \phi))$
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Sine-Cosine Product-Sum	$\sinh(\theta)\cosh(\phi) = \frac{1}{2}(\sinh(\theta + \phi) + \sinh(\theta - \phi))$
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Tangent Sum/Difference	$\tanh(\theta \pm \phi) = \frac{\tanh(\theta) \pm \tanh(\phi)}{1 \pm \tanh(\theta)\tanh(\phi)}$
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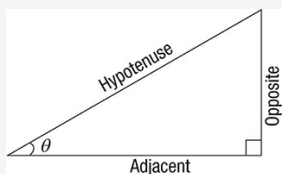
Tangent Sum	$\tanh(\theta) \pm \tanh(\phi) = \frac{\sinh(\theta) \pm \sinh(\phi)}{\cosh(\theta) \pm \cosh(\phi)}$
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Tangent Product	$\tanh(\theta)\tanh(\phi) = \frac{\sinh(\theta)\sinh(\phi)}{\cosh(\theta)\cosh(\phi)}$
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Tangent-Cotangent Product	$\tanh(\theta)\operatorname{coth}(\phi) = \frac{\tanh(\theta) + \operatorname{coth}(\phi)}{\operatorname{coth}(\theta) + \tanh(\phi)}$
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Right-Triangle Relations



$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} \quad \csc \theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} \quad \sec \theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}} \quad \cot \theta = \frac{\text{adjacent}}{\text{opposite}}$$

Hyperbolic-Inverse Reciprocal Identities

Sine Reciprocal	$\operatorname{arcsinh}(1/x) = \operatorname{arccsch}(x)$
Cosine Reciprocal	$\operatorname{arccosh}(1/x) = \operatorname{arcsech}(x)$
Tangent Reciprocal	$\operatorname{arctanh}(1/x) = \operatorname{arcoth}(x)$
Cosecant Reciprocal	$\operatorname{arccsch}(1/x) = \operatorname{arcsinh}(x)$
Secant Reciprocal	$\operatorname{arcsech}(1/x) = \operatorname{arccosh}(x)$
Cotangent Reciprocal	$\operatorname{arcoth}(1/x) = \operatorname{arctanh}(x)$

(HI) Negative Input Identities

Inverse Sine Odd	$\operatorname{arcsinh}(-x) = -\operatorname{arcsinh}(x)$
Inverse Tangent Odd	$\operatorname{arctanh}(-x) = -\operatorname{arctanh}(x)$
Inverse Cosecant Odd	$\operatorname{arccsch}(-x) = -\operatorname{arccsch}(x)$
Inverse Cotangent Odd	$\operatorname{arcoth}(-x) = -\operatorname{arcoth}(x)$

(HI) Half/Multiple Substitution Identities

Half Sine Substitution	$\frac{1}{2}\operatorname{arcsinh}(x) = \pm \operatorname{arcsinh}(\sqrt{\frac{1+x^2}{1-x^2}})$
Half Cosine Substitution	$\frac{1}{2}\operatorname{arccosh}(x) = \operatorname{arccosh}(\sqrt{\frac{x+1}{2}})$
Half Tangent Substitution	$\frac{1}{2}\operatorname{arctanh}(x) = \operatorname{arctanh}(x/(1+\sqrt{1-x^2}))$
Double Sine Substitution 1	$2\operatorname{arcsinh}(x) = \operatorname{arcsinh}(2x\sqrt{1+x^2})$
Double Sine Substitution 2	$2\operatorname{arcsinh}(x) = \pm \operatorname{arccosh}(2x^2+1)$
Double Cosine Substitution	$2\operatorname{arccosh}(x) = \operatorname{arccosh}(2x^2-1), x \geq 1$
Double Tangent Substitution	$2\operatorname{arctanh}(x) = \operatorname{arctanh}(2x/(1+x^2)), x < 1$
Triple Sine Substitution	$3\operatorname{arcsinh}(x) = \operatorname{arcsinh}(3x+4x^3)$
Triple Cosine Substitution	$3\operatorname{arccosh}(x) = \operatorname{arccosh}(4x^3-3x)$
Triple Tangent Substitution	$3\operatorname{arctanh}(x) = \operatorname{arctanh}(h((3x+x^3)/(1+3x^2)))$

(HI) Sum/Difference Identities

Sine Sum/Difference	$\operatorname{arcsinh}(x) \pm \operatorname{arcsinh}(y) = \operatorname{arcsinh}(x\sqrt{y^2+1} \pm y\sqrt{x^2+1})$
Cosine Sum/Difference	$\operatorname{arccosh}(x) \pm \operatorname{arccosh}(y) = \operatorname{arccosh}(xy \pm \sqrt{(x^2-1)(y^2-1)})$
Sine-Cosine Sum/Difference 1	$\operatorname{arcsinh}(x) \pm \operatorname{arccosh}(y) = \operatorname{arcsinh}(xy \pm \sqrt{(x^2+1)(y^2-1)})$
Sine-Cosine Sum/Difference 2	$\operatorname{arcsinh}(x) \pm \operatorname{arccosh}(y) = \pm \operatorname{arccosh}(y\sqrt{x^2+1} \pm x\sqrt{y^2-1})$
Tangent Sum/Difference	$\operatorname{arctanh}(x) \pm \operatorname{arctanh}(y) = \operatorname{arctanh}((x \pm y)/(1 \pm xy))$

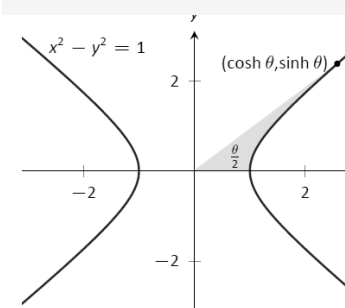
(HI) Logarithmic/Compositional Conversions

Sine Logarithmic	$\ln(x) = \operatorname{arcsinh}((x^2-1)/2x), x > 0$
Cosine Logarithmic	$\ln(x) = \pm \operatorname{arccosh}((x^2+1)/2x)$
Tangent Logarithmic	$\ln(x) = \operatorname{arctanh}((x^2-1)/(x^2+1))$
Sine Hyperbolic-Circular 1	$\operatorname{arcsinh}(\tan(x)) = \ln((\sec(x+\pi n) + \tan(x+\pi n)))$
Sine Hyperbolic-Circular 2	$\operatorname{arcsinh}(\tan(x)) = \pm \operatorname{arccosh}(\sec(x+\pi n))$

(HI) Logarithmic/Compositional Conversions (cont)

Tangent Hyperbolic-Circular 1	$\operatorname{arctanh}(\cos(-2x)) = \ln(\cot(x))$
Tangent Hyperbolic-Circular 2	$\pm \operatorname{arctanh}(\sin(x)) = \pm \operatorname{arcsinh}(\tan(x))$
Sine Cofunctional 1	$\operatorname{arcsinh}(x) = \operatorname{arctanh}(x/\sqrt{1+x^2})$
Sine Cofunctional 2	$\operatorname{arcsinh}(x) = \pm \operatorname{arccosh}(\sqrt{1+x^2})$
Cosine Cofunctional 1	$\operatorname{arccosh}(x) = \operatorname{arcsinh}(\sqrt{x^2-1}) , x \geq 1$
Cosine Cofunctional 2	$\operatorname{arccosh}(x) = \operatorname{arctanh}(\sqrt{x^2-1}/x) , x \geq 1$
Tangent Cofunctional 1	$\operatorname{arctanh}(x) = \operatorname{arcsinh}(x/\sqrt{1-x^2})$
Tangent Cofunctional 2	$\operatorname{arctanh}(x) = \pm \operatorname{arccosh}(1/\sqrt{1-x^2})$

Unit Hyperbola



$$x^2 - y^2 = 1$$

