

Circular Functions Definitions

Name	Right-Triangle Definition	Domain	Range
Sine Function	$\sin(\theta) = o/h$	$(-\infty, \infty)$	$[-1, 1]$
Cosine Function	$\cos(\theta) = a/h$	$(-\infty, \infty)$	$[-1, 1]$
Tangent Function	$\tan(\theta) = o/a$	$\{\theta \theta \neq \pi/2 \pm \pi n\}$	$(-\infty, \infty)$
Cosecant Function	$\csc(\theta) = h/o$	$\{\theta \theta \neq \pi n\}$	$(-\infty, -1] \cup [1, \infty)$
Secant Function	$\sec(\theta) = h/a$	$\{\theta \theta \neq \pi/2 \pm \pi n\}$	$(-\infty, -1] \cup [1, \infty)$
Cotangent Function	$\cot(\theta) = a/o$	$\{\theta \theta \neq \pi n\}$	$(-\infty, \infty)$
Inverse Sine Function	$\arcsin(o/h) = \theta$	$[-1, 1]$	$[-\pi/2, \pi/2]$
Inverse Cosine Function	$\arccos(a/h) = \theta$	$[-1, 1]$	$[0, \pi]$
Inverse Tangent Function	$\arctan(o/a) = \theta$	$(-\infty, \infty)$	$(-1, 1)$
Inverse Cosecant Function	$\operatorname{arccsc}(h/o) = \theta$	$(-\infty, -1) \cup (1, \infty)$	$[-\pi/2, 0) \cup (0, \pi/2]$
Inverse Secant Function	$\operatorname{arcsec}(h/a) = \theta$	$(-\infty, -1) \cup (1, \infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$
Inverse Cotangent Function	$\operatorname{arccot}(a/o) = \theta$	$(-\infty, \infty)$	$(0, 1)$
Circular Euler Relation	$e^{\pm i\theta} = \cos(\theta) \pm i\sin(\theta)$		
De Moivre's Theorem	$e^{in\theta} = (\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$		

$n \in \mathbb{N}_1 = \{1, 2, 3, 4, 5, \dots\}$

"h" is the "hypotenuse" leg of a right triangle. It is directly across the right (90°) angle, and it has the longest length of the three sides.

"o" is the "opposite" leg of a right triangle. It is directly across the chosen angle θ .

"a" is the "adjacent" leg of a right triangle. It is the leg that is neither the hypotenuse leg, nor the opposite leg.

By the Pythagorean theorem, $o^2 + a^2 = h^2$

Hyperbolic Functions Definitions

Name	Exponential Definition	Domain	Range
Hyperbolic Sine Function	$\sinh(\theta) = (e^\theta - e^{-\theta})/2$	$(-\infty, \infty)$	$(-\infty, \infty)$
Hyperbolic Cosine Function	$\cosh(\theta) = (e^\theta + e^{-\theta})/2$	$(-\infty, \infty)$	$[1, \infty)$
Hyperbolic Tangent Function	$\tanh(\theta) = (e^\theta - e^{-\theta})/(e^\theta + e^{-\theta})$	$(-\infty, \infty)$	$(-1, 1)$
Hyperbolic Cosecant Function	$\operatorname{csch}(\theta) = 2/(e^\theta - e^{-\theta})$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
Hyperbolic Secant Function	$\operatorname{sech}(\theta) = 2/(e^\theta + e^{-\theta})$	$(-\infty, \infty)$	$(0, 1]$



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Hyperbolic Functions Definitions (cont)

Hyperbolic Cotangent Function	$\coth(\theta) = (e^\theta + e^{-\theta}) / (e^\theta - e^{-\theta})$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, -1) \cup (1, \infty)$
Inverse Hyperbolic Sine Function	$\operatorname{arcsinh}(x) = \ln(x + \sqrt{x^2 + 1})$	$(-\infty, \infty)$	$(-\infty, \infty)$
Inverse Hyperbolic Cosine Function	$\operatorname{arcosh}(x) = \ln(x + \sqrt{x^2 - 1})$	$[1, \infty)$	$[0, \infty)$
Inverse Hyperbolic Tangent Function	$\operatorname{artanh}(x) = \frac{1}{2} \ln((1+x)/(1-x))$	$(-1, 1)$	$(-\infty, \infty)$
Inverse Hyperbolic Cosecant Function	$\operatorname{arccsch}(x) = \ln((1 \pm \sqrt{1+x^2})/x)$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
Inverse Hyperbolic Secant Function	$\operatorname{arcsech}(x) = \ln((1 + \sqrt{1-x^2})/x)$	$(0, 1]$	$[0, \infty)$
Inverse Hyperbolic Cotangent Function	$\operatorname{arccoth}(x) = \frac{1}{2} \ln((x+1)/(x-1))$	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, 0) \cup (0, \infty)$
Hyperbolic Euler Relation	$e^{\pm\theta} = \cosh(\theta) \pm \sinh(\theta)$		
De Moivre's Theorem (Hyperbolic)	$e^{n\theta} = (\cosh(\theta) + \sinh(\theta))^n = \cosh(n\theta) + \sinh(n\theta)$		
$n \in \mathbb{N}_1 = \{1, 2, 3, 4, 5, \dots\}$			

Complex Definitions

Name	Complex Relation	Circular-Hyperbolic Relation
Complex Sine	$\sin(z) = (e^{iz} - e^{-iz}) / 2i$	$\sin(z) = -i \sinh(iz)$
Complex Cosine	$\cos(z) = (e^{iz} + e^{-iz}) / 2$	$\cos(z) = \cosh(iz)$
Complex Tangent	$\tan(z) = i(e^{iz} - e^{-iz}) / (e^{iz} + e^{-iz})$	$\tan(z) = i \tanh(iz)$
Complex Cosecant	$\csc(z) = 2i / (e^{iz} - e^{-iz})$	$\csc(z) = i \operatorname{csch}(iz)$
Complex Secant	$\sec(z) = 2 / (e^{iz} + e^{-iz})$	$\sec(z) = \operatorname{sech}(iz)$
Complex Cotangent	$\cot(z) = i(e^{iz} + e^{-iz}) / (e^{iz} - e^{-iz})$	$\cot(z) = i \coth(iz)$
Complex Inverse Sine	$\arcsin(z) = -i \ln(iz \pm \sqrt{1-z^2})$	$\arcsin(z) = -i \operatorname{arcsinh}(iz)$
Complex Inverse Cosine	$\arccos(z) = -i \ln(z \pm i \sqrt{1-z^2})$	$\arccos(z) = \pm i \operatorname{arcosh}(z)$
Complex Inverse Tangent	$\arctan(z) = (i/2) \ln((i+z)/(i-z))$	$\arctan(z) = i \operatorname{arctanh}(z)$
Complex Inverse Cosecant	$\operatorname{arccsc}(z) = -i \ln((i + \sqrt{z^2 - 1})/z)$	$\operatorname{arccsc}(z) = i \operatorname{arccsch}(z)$
Complex Inverse Secant	$\operatorname{arcsec}(z) = -i \ln((1 + \sqrt{1-z^2})/z)$	$\operatorname{arcsec}(z) = \pm i \operatorname{arcsech}(z)$
Complex Inverse Cotangent	$\operatorname{arccot}(z) = -(i/2) \ln((z+i)/(z-i))$	$\operatorname{arccot}(z) = \pm i \operatorname{arccoth}(z)$
Complex Hyperbolic Sine	None	$\sinh(z) = -i \sin(iz)$



Complex Definitions (cont)

Complex Hyperbolic Cosine	None	$\cosh(z) = \cos(iz)$
Complex Hyperbolic Tangent	None	$\tanh(z) = -i \tan(iz)$
Complex Hyperbolic Cosecant	None	$\operatorname{csch}(z) = i \operatorname{csc}(iz)$
Complex Hyperbolic Secant	None	$\operatorname{sech}(z) = \sec(iz)$
Complex Hyperbolic Cotangent	None	$\operatorname{coth}(z) = i \cot(iz)$
Complex Inverse Hyperbolic Sine	None	$\operatorname{arcsinh}(z) = -i \arcsin(iz)$
Complex Inverse Hyperbolic Cosine	None	$\operatorname{arccosh}(z) = \pm i \arccos(z)$
Complex Inverse Hyperbolic Tangent	None	$\operatorname{arctanh}(z) = -i \arctan(iz)$
Complex Inverse Hyperbolic Cosecant	None	$\operatorname{arccsch}(z) = -i \operatorname{arccsc}(iz)$
Complex Inverse Hyperbolic Secant	None	$\operatorname{arcsech}(z) = \pm i \operatorname{arcsec}(z)$
Complex Inverse Hyperbolic Cotangent	None	$\operatorname{arccoth}(z) = -i \operatorname{arccot}(iz)$

$i = \sqrt{-1}$

z is a complex variable of the form $a+bi$, where a and b are real numbers, and i is the imaginary unit

Circular Functions Unit Circle Values

θ (Radians)	θ (Degrees)	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$	$\csc(\theta)$	$\sec(\theta)$	$\cot(\theta)$
0	0°	0	1	0	undefined	1	undefined
$\pi/6$	30°	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$	2	$2\sqrt{3}/3$	$\sqrt{3}$
$\pi/4$	45°	$\sqrt{2}/2$	$\sqrt{2}/2$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\pi/3$	60°	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$	$2\sqrt{3}/3$	2	$\sqrt{3}/3$
$\pi/2$	90°	1	0	undefined	1	undefined	0
$2\pi/3$	120°	$\sqrt{3}/2$	$-1/2$	$-\sqrt{3}$	$2\sqrt{3}/3$	-2	$-\sqrt{3}/3$
$3\pi/4$	135°	$\sqrt{2}/2$	$-\sqrt{2}/2$	-1	$\sqrt{2}$	$-\sqrt{2}$	-1
$5\pi/6$	150°	$1/2$	$-\sqrt{3}/2$	$-\sqrt{3}/3$	2	$-2\sqrt{3}/3$	$-\sqrt{3}$
π	180°	0	-1	0	undefined	-1	undefined
$7\pi/6$	210°	$-1/2$	$-\sqrt{3}/2$	$\sqrt{3}/3$	-2	$-2\sqrt{3}/3$	$\sqrt{3}$
$5\pi/4$	225°	$-\sqrt{2}/2$	$-\sqrt{2}/2$	1	$-\sqrt{2}$	$-\sqrt{2}$	1
$4\pi/3$	240°	$-\sqrt{3}/2$	$-1/2$	$\sqrt{3}$	$-2\sqrt{3}/3$	-2	$\sqrt{3}/3$



Circular Functions Unit Circle Values (cont)

$3\pi/2$	270°	-1	0	undefined	-1	undefined	0
$5\pi/3$	300°	$-\sqrt{3}/2$	$1/2$	$-\sqrt{3}$	$-2\sqrt{3}/3$	2	$-\sqrt{3}/3$
$7\pi/4$	315°	$-\sqrt{2}/2$	$\sqrt{2}/2$	-1	$-\sqrt{2}$	$\sqrt{2}$	-1
$11\pi/6$	330°	-1/2	$\sqrt{3}/2$	$-\sqrt{3}/3$	-2	$2\sqrt{3}/3$	$-\sqrt{3}$
2π	360°	0	1	0	undefined	1	undefined

The coordinates $(\cos(\theta), \sin(\theta))$ represent x and y coordinates of θ on the unit circle $x^2+y^2=1$

Circular Compositional Identities

Composition	$\sin(x)$	$\cos(x)$	$\tan(x)$
$\arcsin(x)$	x	$\sqrt{1-x^2}$	$x/\sqrt{1-x^2}$
$\arccos(x)$	$\sqrt{1-x^2}$	x	$\sqrt{1-x^2}/x$
$\arctan(x)$	$x/\sqrt{1+x^2}$	$1/\sqrt{1+x^2}$	x
$\operatorname{arccsc}(x)$	$1/x$	$\sqrt{x^2-1}/ x $	$\pm 1/\sqrt{x^2-1}$
$\operatorname{arcsec}(x)$	$\sqrt{x^2-1}/ x $	$1/x$	$\pm \sqrt{x^2-1}$
$\operatorname{arccot}(x)$	$1/\sqrt{1+x^2}$	$x/\sqrt{1+x^2}$	$1/x$

Each composition is valid on different domains

Hyperbolic Compositional Identities

Composition	$\sinh(x)$	$\cosh(x)$	$\tanh(x)$
$\operatorname{arcsinh}(x)$	x	$\sqrt{1+x^2}$	$x/\sqrt{1+x^2}$
$\operatorname{arcosh}(x)$	$\sqrt{x^2-1}$	x	$\sqrt{x^2-1}/x$
$\operatorname{artanh}(x)$	$x/\sqrt{1-x^2}$	$1/\sqrt{1-x^2}$	x
$\operatorname{arccsch}(x)$	$1/x$	$\sqrt{x^2+1}/ x $	$1/\sqrt{x^2+1}$
$\operatorname{arcsech}(x)$	$\sqrt{1-x^2}/x$	$1/x$	$\sqrt{1-x^2}$
$\operatorname{arcoth}(x)$	$x/\sqrt{1-x^2}$	$ x /\sqrt{x^2-1}$	$1/x$

Each composition is valid on different domains

Circular Quotient & Reciprocal Identities

Cofunctional Phase Shift Properties (cont)

Cosecant	$\csc(\theta) = \sec(\pi/2 - \theta)$
Complimentary	
Cosecant	$\csc(\theta) = \csc(\pi - \theta)$
Supplementary	
Secant Complimentary	$\sec(\theta) = \csc(\pi/2 - \theta)$
Supplementary	$\sec(\theta) = -\sec(\pi - \theta)$
Cotangent	$\cot(\theta) = \tan(\pi/2 - \theta)$
Complimentary	
Cotangent	$\cot(\theta) = -\cot(\pi - \theta)$
Supplementary	

$$n \in \mathbb{N}_1 = \{1, 2, 3, 4, 5, \dots\}$$

Periodicity Properties

Circular Parity Properties

Sine Odd	$\sin(-\theta) = -\sin(\theta)$
Cosine Even	$\cos(-\theta) = \cos(\theta)$
Tangent Odd	$\tan(-\theta) = -\tan(\theta)$
Cosecant Odd	$\csc(-\theta) = -\csc(\theta)$
Secant Even	$\sec(-\theta) = \sec(\theta)$
Cotangent Odd	$\cot(-\theta) = -\cot(\theta)$

Circular Pythagorean Identities

(C) Half/Multiple-Angle Identities (cont)

Tangent Quotient	$\tan(\theta)=\sin(\theta)/\cos(\theta)$
Cotangent Quotient	$\cot(\theta)=\cos(\theta)/\sin(\theta)$
Sine Reciprocal	$\sin(\theta)=1/\csc(\theta)$
Cosine Reciprocal	$\cos(\theta)=1/\sec(\theta)$
Tangent Reciprocal	$\tan(\theta)=1/\cot(\theta)$
Cosecant Reciprocal	$\csc(\theta)=1/\sin(\theta)$
Secant Reciprocal	$\sec(\theta)=1/\cos(\theta)$
Cotangent Reciprocal	$\cot(\theta)=1/\tan(\theta)$

All the following identities are true for values that do not cause division by zero

Cofunctional Phase Shift Properties

Sine Complementary	$\sin(\theta)=\cos(\pi/2-\theta)$
Sine Supplementary	$\sin(\theta)=\sin(\pi-\theta)$
Cosine Complementary	$\cos(\theta)=\sin(\pi/2-\theta)$
Cosine Supplementary	$\cos(\theta)=-\cos(\pi-\theta)$
Tangent Complementary	$\tan(\theta)=\cot(\pi/2-\theta)$
Tangent Supplementary	$\tan(\theta)=-\tan(\pi-\theta)$

Sine Periodicity	$\sin(\theta)=\sin(\theta\pm2\pi n)$
Cosine Periodicity	$\cos(\theta)=\cos(\theta\pm2\pi n)$
Tangent Periodicity	$\tan(\theta)=\tan(\theta\pm\pi n)$
Cosecant Periodicity	$\csc(\theta)=\csc(\theta\pm2\pi n)$
Secant Periodicity	$\sec(\theta)=\sec(\theta\pm2\pi n)$
Cotangent Periodicity	$\cot(\theta)=\cot(\theta\pm\pi n)$
$n \in \mathbb{N}_1 = \{1,2,3,4,5,\dots\}$	

Sine-Cosine Pythagorean	$\sin^2(\theta)+\cos^2(\theta)=1$
Secant-Tangent Pythagorean	$\tan^2(\theta)+1=\sec^2(\theta)$
Cosecant-Cotangent Pythagorean	$1+\cot^2(\theta)=\csc^2(\theta)$
The last two Pythagorean Identities are obtained by dividing all the terms of the original Sine-Cosine Identity by $\cos^2(\theta)$ and $\sin^2(\theta)$, respectively	

(C) Half/Multiple-Angle Identities	
Sine Half-Angle	$\sin(\theta/2)=\pm\sqrt{1/2(1-\cos(\theta))}$
Cosine Half-Angle	$\cos(\theta/2)=\pm\sqrt{1/2(1+\cos(\theta))}$
Tangent Half-Angle 1	$\tan(\theta/2)=\pm\sqrt{(1-\cos(\theta))/(1+\cos(\theta))}$

Tangent Half-Angle 2	$\tan(\theta/2)=(1-\cos(\theta))/\sin(\theta)$
Tangent Half-Angle 3	$\tan(\theta/2)=\sin(\theta)/(1+\cos(\theta))$
Sine Double-Angle 1	$\sin(2\theta)=2\sin(\theta)\cos(\theta)$
Sine Double-Angle 2	$\sin(2\theta)=2\tan(\theta)/(1+\tan^2(\theta))$
Cosine Double-Angle 1	$\cos(2\theta)=\cos^2(\theta)-\sin^2(\theta)$
Cosine Double-Angle 2	$\cos(2\theta)=2\cos^2(\theta)-1$
Cosine Double-Angle 3	$\cos(2\theta)=1-2\sin^2(\theta)$
Cosine Double-Angle 4	$\cos(2\theta)=(1-\tan^2(\theta))/(-1+\tan^2(\theta))$
Tangent Double-Angle 1	$\tan(2\theta)=2\tan(\theta)/(1-\tan^2(\theta))$
Tangent Double-Angle 2	$\tan(2\theta)=2/(\cot(\theta)-\tan(\theta))$
Sine Triple-Angle	$\sin(3\theta)=3\sin(\theta)-4\sin^3(\theta)$



(C) Half/Multiple-Angle Identities (cont)

Cosine Triple-Angle $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$

Tangent Triple-Angle $\tan(3\theta) = (3\tan(\theta) - \tan^3(\theta)) / (1 - 3\tan^2(\theta))$

Sine Multiple-Angle Formula:
 $\sin(n\theta) = \sum_{k=0}^n \binom{n}{k} \cos^k(\theta) \sin^{n-k}(\theta) \sin((\pi/2)(n-k))$
 Cosine Multiple-Angle Formula:
 $\cos(n\theta) = \sum_{k=0}^n \binom{n}{k} \cos^k(\theta) \sin^{n-k}(\theta) \cos((\pi/2)(n-k))$
 All the following identities are true for values that do not cause division by zero

Circular Sum/Difference/Product Identities

Sine Sum/Difference $\sin(\theta \pm \phi) = \sin(\theta) \cos(\phi) \pm \cos(\theta) \sin(\phi)$

Sine Sum-Product $\sin(\theta) \pm \sin(\phi) = 2\sin((\theta \pm \phi)/2) \cos((\theta \mp \phi)/2)$

Sine Product-Sum $\sin(\theta) \sin(\phi) = \frac{1}{2}(\cos(\theta - \phi) - \cos(\theta + \phi))$

Cosine Sum/Difference $\cos(\theta \pm \phi) = \cos(\theta) \cos(\phi) \mp \sin(\theta) \sin(\phi)$

Cosine Sum-Product $\cos(\theta) \pm \cos(\phi) = 2\cos((\theta \pm \phi)/2) \cos((\theta \mp \phi)/2)$

Circular Sum/Difference/Product Identities (cont)

Cosine Product-Sum $\cos(\theta) \cos(\phi) = \frac{1}{2}(\cos(\theta - \phi) + \cos(\theta + \phi))$

Sine-Cosine Product-Sum $\sin(\theta) \cos(\phi) = \frac{1}{2}(\sin(\theta - \phi) + \sin(\theta + \phi))$

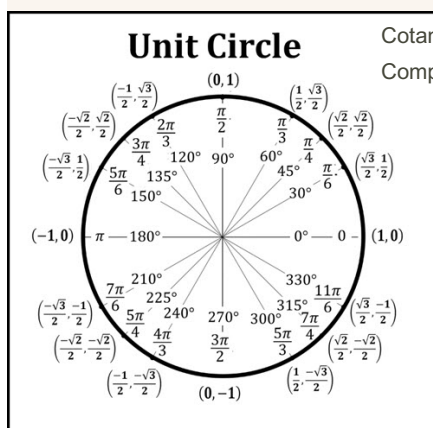
Tangent Sum/Difference $\tan(\theta \pm \phi) = (\tan(\theta) \pm \tan(\phi)) / (1 \mp \tan(\theta) \tan(\phi))$

Tangent Sum $\tan(\theta) \pm \tan(\phi) = \sin(\theta \pm \phi) / (\cos(\theta) \cos(\phi))$

Tangent Product $\tan(\theta) \tan(\phi) = (\tan(\theta) + \tan(\phi)) / (\cot(\theta) + \cot(\phi))$

Tangent-Cotangent Product $\tan(\theta) \cot(\phi) = (\tan(\theta) + \cot(\phi)) / (\cot(\theta) + \tan(\phi))$

Sine and Cosine Unit Circle



$$x^2 + y^2 = 1$$

Circular-Inverse Reciprocal Identities

Sine Reciprocal $\arcsin(1/x) = \arccsc(x)$

Cosine Reciprocal $\arccos(1/x) = \arcsec(x)$

Tangent Reciprocal 1 $\arctan(1/x) = \text{arccot}(x), x > 0$

Tangent Reciprocal 2 $\arctan(1/x) = \text{arccot}(x) - \pi, x < 0$

Cosecant Reciprocal $\text{arccsc}(1/x) = \arcsin(x)$

Secant Reciprocal $\text{arcsec}(1/x) = \arccos(x)$

Cotangent Reciprocal 1 $\text{arccot}(1/x) = \arctan(x), x > 0$

Cotangent Reciprocal 2 $\text{arccot}(1/x) = \arctan(x) + \pi, x < 0$

Circular-Inverse Complimentary Identities

Sine Complimentary $\arcsin(x) = \pi/2 - \arccos(x)$

Cosine Complimentary $\arccos(x) = \pi/2 - \arcsin(x)$

Tangent Complimentary $\arctan(x) = \pi/2 - \text{arccot}(x)$

Cosecant Complimentary $\text{arccsc}(x) = \pi/2 - \text{arcsec}(x)$

Secant Complimentary $\text{arcsec}(x) = \pi/2 - \text{arccsc}(x)$

Cotangent Complimentary $\text{arccot}(x) = \pi/2 - \arctan(x)$

Circular-Inverse Negative Input Identities

Sine Odd $\arcsin(-x) = -\arcsin(x)$

Cosine Translation $\arccos(-x) = \pi - \arccos(x)$

Tangent Odd $\arctan(-x) = -\arctan(x)$

Cosecant Odd $\text{arccsc}(-x) = -\text{arccsc}(x)$

Secant Translation $\text{arcsec}(-x) = \pi - \text{arcsec}(x)$

Cotangent Translation $\text{arccot}(-x) = \pi - \text{arccot}(x)$

(CI) Half/Multiple Substitution Identities

Half Sine Substitution 1 $\frac{1}{2}\arcsin(x) = \arcsin(\sqrt{(1+x)/2}) - \pi/4$

Half Sine Substitution 2 $\frac{1}{2}\arcsin(x) = \pi/4 - \arcsin(\sqrt{(1-x)/2})$

Half Cosine Substitution 1 $\frac{1}{2}\arccos(x) = \arccos(\sqrt{(1+x)/2})$

Half Cosine Substitution 2 $\frac{1}{2}\arccos(x) = \pi/2 - \arccos(\sqrt{(1-x)/2})$

Double Sine Substitution $2\arcsin(x) = \arcsin(2x\sqrt{1-x^2}), |x| \leq \pi/2$

Double Cosine Substitution 1 $2\arccos(x) = \arccos(2x^2 - 1), x \geq 0$



(CI) Half/Multiple Substitution Identities (cont)

Double Cosine Substitution 2

$$2\arccos(x) = 2\pi - \arccos(2x^2 - 1), \quad x \leq 0$$

Double Tangent Substitution 1

$$2\arctan(x) = \arcsin\left(\frac{2x}{1+x^2}\right), \quad |x| \leq 1$$

Double Tangent Substitution 2

$$2\arctan(x) = \pm \arccos\left(\frac{1-x^2}{1+x^2}\right)$$

Double Tangent Substitution 3

$$2\arctan(x) = \arctan\left(\frac{2x}{1-x^2}\right), \quad |x| < 1$$

Triple Sine Substitution 1

$$3\arcsin(x) = \arcsin(3x - 4x^3), \quad |x| \leq \frac{1}{2}$$

Triple Sine Substitution 2

$$3\arcsin(x) = \arcsin(4x^3 - 3x) \pm \pi, \quad |x| \geq \frac{1}{2}$$

Triple Cosine Substitution 1

$$3\arccos(x) = \arccos(3x - 4x^3), \quad |x| \leq \frac{1}{2}$$

Triple Cosine Substitution 2

$$3\arccos(x) = \arccos(4x^3 - 3x) \pm \pi, \quad |x| \geq \frac{1}{2}$$

Triple Tangent Substitution 1

$$3\arctan(x) = \arctan\left(\frac{3x - x^3}{1 - 3x^2}\right), \quad |x| \leq \sqrt{3}/3$$

Triple Tangent Substitution 2

$$3\arctan(x) = \arctan\left(\frac{3x - x^3}{1 - 3x^2}\right) \pm \pi, \quad |x| \geq \sqrt{3}/3$$

Circular-Inverse Sum/Difference Identities

Sine Sum/Difference

$$\arcsin(x) \pm \arcsin(y) = \arcsin(x\sqrt{1-y^2} \pm y\sqrt{1-x^2})$$

Cosine Sum/Difference

$$\arccos(x) \pm \arccos(y) = \arccos(xy \mp \sqrt{1-x^2}\sqrt{1-y^2})$$

Cosine-Sine Sum/Difference

$$\arccos(x) \pm \arcsin(y) = \arccos(x\sqrt{1-y^2} \mp y\sqrt{1-x^2})$$

Tangent Sum/Difference

$$\arctan(x) \pm \arctan(y) = \arctan\left(\frac{x \pm y}{1 \mp xy}\right), \quad 1 \mp xy \neq 0$$

Law of Sines/Cosines/Tangents

Law of Sines 1

$$\sin(\alpha)/a = \sin(\beta)/b = \sin(\gamma)/c$$

Law of Sines 2

$$a/\sin(\alpha) = b/\sin(\beta) = c/\sin(\gamma)$$

Law of Cosines 1

$$a^2 = b^2 + c^2 - 2bc\cos(\alpha)$$

Law of Cosines 2

$$b^2 = a^2 + c^2 - 2ac\cos(\beta)$$

Law of Cosines 3

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma)$$

Law of Tangents 1

$$(a-b)/(a+b) = \tan((\alpha-\beta)/2)/\tan((\alpha+\beta)/2)$$

Law of Sines/Cosines/Tangents (cont)

Law of Tangents 2

$$(b-c)/(b+c) = \tan((\beta-\gamma)/2)/\tan((\beta+\gamma)/2)$$

Law of Tangents 3

$$(c-a)/(c+a) = \tan((\gamma-\alpha)/2)/\tan((\gamma+\alpha)/2)$$

Side lengths a, b, and c are opposite of the angles α , β , and γ , respectively.

Measurements And Formulas

Radians-Degrees

$$1 \text{ radian} = 180/\pi \text{ degrees}; \quad 1^\circ = (180/\pi)^\circ$$

Degrees-Radians

$$1 \text{ degree} = \pi/180 \text{ radians}; \quad 1^\circ = \pi/180 \text{ radians}$$

Degrees, Minutes, and Seconds (DMS)

$$1 \text{ degree} = 60 \text{ minutes} = 3600 \text{ seconds}; \quad 1^\circ = 60' = 3600''$$

Arc Length/Angular Displacement

$$s = r\theta \text{ units}$$

Sector Area

$$\frac{1}{2}r^2\theta \text{ units}^2$$

Area of a Triangle

$$A_T = \frac{1}{2}bh \text{ units}^2$$

Area of a Circle

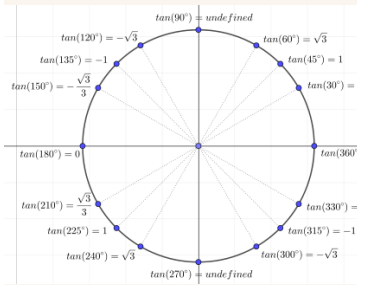
$$A_C = \pi r^2 \text{ units}^2$$

Pythagorean Theorem

$$a^2 + b^2 = c^2$$

Radians are unitless
a, b, and c are side lengths of a right-triangle

Tangent Unit Circle



(H) Quotient & Reciprocal Identities

Tangent Quotient

$$\tanh(\theta) = \sinh(-\theta)/\cosh(\theta)$$

Cotangent Quotient

$$\coth(\theta) = \cosh(-\theta)/\sinh(\theta)$$

Sine Reciprocal

$$\sinh(\theta) = 1/\cosh(\theta)$$

Cosine Reciprocal

$$\cosh(\theta) = 1/\sinh(\theta)$$

Tangent Reciprocal

$$\tanh(\theta) = 1/\coth(\theta)$$

Cosecant Reciprocal

$$\csch(\theta) = 1/\sinh(\theta)$$

Secant Reciprocal

$$\operatorname{sech}(\theta) = 1/\cosh(\theta)$$

Cotangent Reciprocal

$$\coth(\theta) = 1/\tanh(\theta)$$

All the following identities are true for values that do not cause division by zero

Hyperbolic Parity Properties	
Sine Odd	$\sinh(-\theta) = -\sinh(\theta)$
Cosine Even	$\cosh(-\theta) = \cosh(\theta)$
Tangent Odd	$\tanh(-\theta) = -\tanh(\theta)$
Cosecant Odd	$\operatorname{csch}(-\theta) = -\operatorname{csch}(\theta)$
Secant Even	$\operatorname{sech}(-\theta) = \operatorname{sech}(\theta)$
Cotangent Odd	$\operatorname{coth}(-\theta) = -\operatorname{coth}(\theta)$

Hyperbolic Pythagorean Identities	
Sine-Cosine Pythagorean	$\cosh^2(\theta) - \sinh^2(\theta) = 1$
Secant Pythagorean	$1 - \tanh^2(\theta) = \operatorname{sech}^2(\theta)$
Cosecant Pythagorean	$\operatorname{coth}^2(\theta) - 1 = \operatorname{csch}^2(\theta)$

The last two Hyperbolic Pythagorean Identities are obtained by dividing all the terms of the original Sine-Cosine Identity by $\cosh^2(\theta)$ and $\sinh^2(\theta)$, respectively

(H) Half-Angle & Multiple-Angle Identities	
Sine Half-Angle	$\sinh(\theta/2) = \pm \sqrt{\frac{1}{2}(\cosh(\theta) - 1)}$
Cosine Half-Angle	$\cosh(\theta/2) = \sqrt{\frac{1}{2}(\cosh(\theta) + 1)}$

(H) Half-Angle & Multiple-Angle Identities (cont)	
Tangent Half-Angle 1	$\tanh(\theta/2) = \pm \sqrt{\frac{\cosh(\theta) - 1}{\cosh(\theta) + 1}}$
Tangent Half-Angle 2	$\tanh(\theta/2) = \frac{\cosh(\theta) - 1}{\sinh(\theta)}$
Tangent Half-Angle 3	$\tanh(\theta/2) = \frac{\sinh(\theta)}{\cosh(\theta) + 1}$

Sine Double-Angle 1	$\sinh(2\theta) = 2\sinh(\theta)\cosh(\theta)$
Sine Double-Angle 2	$\sinh(2\theta) = 2\tanh(\theta)/(1 - \tanh^2(\theta))$
Cosine Double-Angle 1	$\cosh(2\theta) = \cosh^2(\theta) + \sinh^2(\theta)$
Cosine Double-Angle 2	$\cosh(2\theta) = 2\cosh^2(\theta) - 1$
Cosine Double-Angle 3	$\cosh(2\theta) = -1 + 2\sinh^2(\theta)$
Cosine Double-Angle 4	$\cosh(2\theta) = (1 + \tanh^2(\theta))/(1 - \tanh^2(\theta))$
Tangent Double-Angle 1	$\tanh(2\theta) = 2\tanh(\theta)/(1 + \tanh^2(\theta))$
Tangent Double-Angle 2	$\tanh(2\theta) = 2/(\operatorname{coth}(\theta) + \tanh(\theta))$

(H) Half-Angle & Multiple-Angle Identities (cont)	
Sine Triple-Angle	$\sinh(3\theta) = 3\sinh(\theta) + 4\sinh^3(\theta)$
Cosine Triple-Angle	$\cosh(3\theta) = 4\cosh^3(\theta) - 3\cosh(\theta)$
Tangent Triple-Angle	$\tanh(3\theta) = \frac{3\tanh(\theta) + \tanh^3(\theta)}{1 + 3\tanh^2(\theta)}$

(H) Sum/Difference/Product Identities		
Sine Sum/Difference	$\sinh(\theta \pm \phi)$	$\sinh(\theta) \pm \cosh(\phi) \pm \sinh(\theta)\sinh(\phi)$

Sine Sum-Product	$\sinh(\theta) \pm \sinh(\phi)$	$2\sinh((\theta \pm \phi)/2)\cosh((\theta \mp \phi)/2)$
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Sine Product-Sum	$\sinh(\theta)\sinh(\phi)$	$\frac{1}{2}(\cosh(\theta + \phi) - \cosh(\theta - \phi))$
Cosine Sum/Difference	$\cosh(\theta \pm \phi)$	$\cosh(\theta)\cosh(\phi) \mp \sinh(\theta)\sinh(\phi)$

Cosine Sum-Product 1	$\cosh(\theta) + \cosh(\phi)$	$2\cosh((\theta + \phi)/2)\cosh((\theta - \phi)/2)$
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(H) Sum/Difference/Product Identities (cont)		
Cosine Sum-Product 2	$\cosh(\theta) - \cosh(\phi)$	$2\sinh((\theta - \phi)/2)\sinh((\theta + \phi)/2)$
Cosine Product-Sum	$\cosh(\theta)\cosh(\phi)$	$\frac{1}{2}(\cosh(\theta + \phi) + \cosh(\theta - \phi))$

Sine-Cosine Product-Sum	$\sinh(\theta)\cosh(\phi)$	$\frac{1}{2}(\sinh(\theta + \phi) + \sinh(\theta - \phi))$
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Tangent Sum/Difference	$\tanh(\theta \pm \phi)$	$\frac{\tanh(\theta) \pm \tanh(\phi)}{1 \pm \tanh(\theta)\tanh(\phi)}$
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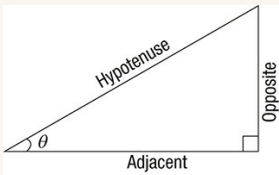
Tangent Sum	$\tanh(\theta) \pm \tanh(\phi)$	$\frac{\sinh(\theta \pm \phi)}{\cosh(\theta) \pm \cosh(\phi)}$
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Tangent Product	$\tanh(\theta)\tanh(\phi)$	$\frac{\tanh(\theta) + \tanh(\phi)}{\operatorname{coth}(\theta) + \operatorname{coth}(\phi)}$
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Tangent-Cotangent Product	$\tanh(\theta)\operatorname{coth}(\phi)$	$\frac{\tanh(\theta) + \operatorname{coth}(\phi)}{\operatorname{coth}(\theta) + \tanh(\phi)}$
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Right-Triangle Relations



$$\begin{aligned}\sin \theta &= \frac{\text{opposite}}{\text{hypotenuse}} & \csc \theta &= \frac{\text{hypotenuse}}{\text{opposite}} \\ \cos \theta &= \frac{\text{adjacent}}{\text{hypotenuse}} & \sec \theta &= \frac{\text{hypotenuse}}{\text{adjacent}} \\ \tan \theta &= \frac{\text{opposite}}{\text{adjacent}} & \cot \theta &= \frac{\text{adjacent}}{\text{opposite}}\end{aligned}$$

Hyperbolic-Inverse Reciprocal Identities

Sine Reciprocal	$\operatorname{arcsinh}(1/x) = \operatorname{arccsch}(x)$
Cosine Reciprocal	$\operatorname{arccosh}(1/x) = \operatorname{arcsech}(x)$
Tangent Reciprocal	$\operatorname{arctanh}(1/x) = \operatorname{arccoth}(x)$
Cosecant Reciprocal	$\operatorname{arccsch}(1/x) = \operatorname{arcsinh}(x)$
Secant Reciprocal	$\operatorname{arcsech}(1/x) = \operatorname{arccosh}(x)$
Cotangent Reciprocal	$\operatorname{arccoth}(1/x) = \operatorname{arctanh}(x)$

(HI) Negative Input Identities

Inverse Sine Odd	$\operatorname{arcsinh}(-x) = -\operatorname{arcsinh}(x)$
Inverse Tangent Odd	$\operatorname{arctanh}(-x) = -\operatorname{arctanh}(x)$
Inverse Cosecant Odd	$\operatorname{arccsch}(-x) = -\operatorname{arccsch}(x)$
Inverse Cotangent Odd	$\operatorname{arccoth}(-x) = -\operatorname{arccoth}(x)$

(HI) Half/Multiple Substitution Identities

Half Substitution	$\frac{1}{2}\operatorname{arcsinh}(x) = \pm \operatorname{arcsinh}(\sqrt{\frac{1}{2}((1+x^2)-1)/2})$
Half Cosine Substitution	$\frac{1}{2}\operatorname{arccosh}(x) = \operatorname{arccosh}(\sqrt{\frac{1}{2}((x+1)/2)})$
Half Tangent Substitution	$\frac{1}{2}\operatorname{arctanh}(x) = \operatorname{arctanh}(x/(1+\sqrt{1-x^2}))$
Double Sine Substitution 1	$2\operatorname{arcsinh}(x) = \operatorname{arcsinh}(2x\sqrt{1+x^2})$
Double Sine Substitution 2	$2\operatorname{arcsinh}(x) = \pm \operatorname{arccosh}(2x^2+1)$
Double Cosine Substitution	$2\operatorname{arccosh}(x) = \operatorname{arccosh}(2x^2-1), x \geq 1$
Double Tangent Substitution	$2\operatorname{arctanh}(x) = \operatorname{arctanh}(2x/(1+x^2)), x < 1$
Triple Sine Substitution	$3\operatorname{arcsinh}(x) = \operatorname{arcsinh}(3x+4x^3)$
Triple Cosine Substitution	$3\operatorname{arccosh}(x) = \operatorname{arccosh}(4x^3-3x)$
Triple Tangent Substitution	$3\operatorname{arctanh}(x) = \operatorname{arctanh}((3x+x^3)/(1+3x^2))$

(HI) Sum/Difference Identities

Sine Sum/Difference	$\operatorname{arcsinh}(x) \pm \operatorname{arcsinh}(y) = \operatorname{arcsinh}(x\sqrt{y^2+1} \pm y\sqrt{x^2+1})$
Cosine Sum/Difference	$\operatorname{arccosh}(x) \pm \operatorname{arccosh}(y) = \operatorname{arccosh}(xy \pm \sqrt{(x^2-1)(y^2-1)})$
Sine-Cosine Sum/Difference 1	$\operatorname{arcsinh}(x) \pm \operatorname{arccosh}(y) = \operatorname{arcsinh}(xy \pm \sqrt{(x^2+1)(y^2-1)})$
Sine-Cosine Sum/Difference 2	$\operatorname{arcsinh}(x) \pm \operatorname{arccosh}(y) = \pm \operatorname{arccosh}(y\sqrt{x^2+1} \pm x\sqrt{y^2-1})$
Tangent Sum/Difference	$\operatorname{arctanh}(x) \pm \operatorname{arctanh}(y) = \operatorname{arctanh}((x \pm y)/(1 \pm xy))$

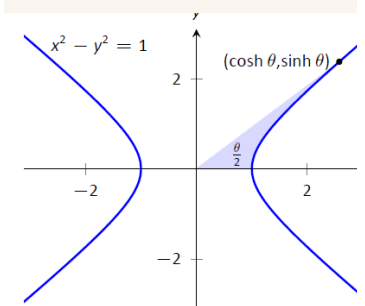
(HI) Logarithmic/Compositional Conversions

Sine Logarithmic	$\ln(x) = \operatorname{arcsinh}((x^2-1)/2x), x > 0$
Cosine Logarithmic	$\ln(x) = \pm \operatorname{arccosh}((x^2+1)/2x)$
Tangent Logarithmic	$\ln(x) = \operatorname{arctanh}((x^2-1)/(x^2+1))$
Sine Hyperbolic-Circular 1	$\operatorname{arcsinh}(\tan(x)) = \ln(\sec(x+\pi n) + \tan(x+\pi n))$
Sine Hyperbolic-Circular 2	$\operatorname{arcsinh}(\tan(x)) = \pm \operatorname{arccosh}(\sec(x+\pi n))$

(HI) Logarithmic/Compositional Conversions (cont)

Tangent Hyperbolic-Circular 1	$\operatorname{arctanh}(\cos(-2x)) = \ln(\cot(x))$
Tangent Hyperbolic-Circular 2	$\pm \operatorname{arctanh}(\sin(x)) = \pm \operatorname{arcsinh}(\tan(x))$
Sine Cofunctional 1	$\operatorname{arcsinh}(x) = \operatorname{arctanh}(x/\sqrt{1+x^2})$
Sine Cofunctional 2	$\operatorname{arcsinh}(x) = \pm \operatorname{arccosh}(\sqrt{1+x^2})$
Cosine Cofunctional 1	$\operatorname{arccosh}(x) = \operatorname{arcsinh}(\sqrt{x^2-1}) , x \geq 1$
Cosine Cofunctional 2	$\operatorname{arccosh}(x) = \operatorname{arctanh}(\sqrt{x^2-1}/x) , x \geq 1$
Tangent Cofunctional 1	$\operatorname{arctanh}(x) = \operatorname{arcsinh}(x/\sqrt{1-x^2})$
Tangent Cofunctional 2	$\operatorname{arctanh}(x) = \pm \operatorname{arccosh}(1/\sqrt{1-x^2})$

Unit Hyperbola



$$x^2 - y^2 = 1$$

