Series

Series Type	General Summation	Convergence	Divergence	Notes
Infinite Series	$\Sigma a_n = \Sigma^{\infty}_{n=k} a_n$, where k is some whole number (k = {0, 1, 2, ...}) for which a_n is well-defined	Varies	Varies	
Harmonic Series	$\Sigma 1/n$	Never converges	Always diverges	The alternating version of this series ($\Sigma(-1)^{n+1}/n$) converges, and $\Sigma 1/n$ is a P-Series with p=1
Geometric Series	$\Sigma^{\infty}_{n=0} ar^n = \Sigma^{\infty}_{n=1} ar^{n-1}$, where a is the first term of the series, and r is the common ratio between terms	Converges if $\|r\|<1$	Diverges if $\|r\|\geq 1$	If the series converges, its sum is $S=a/(1-r)$
P-Series	$\Sigma 1/n^p$, where p is a positive number	Converges if p>1	Diverges if p≤1	
Alternating Series	$\Sigma(-1)^n b_n$, $\Sigma(-1)^{n+1} b_n$, or $\Sigma(-1)^{n-1} b_n$	Converges if $\lim_{n\to\infty} b_n = 0$ and b_n is a decreasing sequence (or eventually decreasing)	Cannot show divergence, inconclusive	
Telescoping Series	$\Sigma(b_n - b_{n+1})$	Varies	Varies	

Alternating Series Estimation Theorem: If $S_n = \Sigma^n_{i=1}(-1)^i b_i$ or $\Sigma^n_{i=1}(-1)^{i-1} b_i$ is the sum of an alternating series that converges, then $|R_n| = |S - S_n| \leq b_{n+1}$

Trigonometric functions like cos(nπ) or sin(nπ+π/2) act as sign alternators, like $(-1)^n$

The Alternating Series Test (AST) checks the limit, but since the AST only concludes convergence, we explicitly apply the Test For Divergence (albeit, redundantly) if the limit fails (even though it checks the same limit)

Series Tests

Test Type	Typical series to use test	Convergence	Divergence	Notes
Test for Divergence	Σa_n	Cannot show convergence, inconclusive	Diverges if $\lim_{n\to\infty} a_n \neq 0$ or DNE	
(Direct) Comparison Test	a_n and b_n are positive-termed ($a_n \geq 0$ and $b_n \geq 0$ for all n) and $a_n \leq b_n$ for all n	Σa_n converges if Σb_n converges	Σb_n diverges if Σa_n diverges	Inconclusive if b_n diverges in proving convergence, or a_n converges in proving divergence

Series Tests (cont)

Limit Comparison Test	a_n and b_n are positive-termed, and lim $_{n\to\infty}$ a_n/b_n=c, where 0<c<∞	Σa_n converges ⟺ Σb_n converges	Σa_n diverges ⟺ Σb_n diverges	Inconclusive if c=0, c=∞, or c DNE
Ratio Test	Σa_n, usually with n! terms, product terms, or $(a_n)^n$	Absolutely converges if lim $_{n\to\infty}$ $\|a_{n+1}/a_n\|<1$	Diverges if lim $_{n\to\infty}$ $\|a_{n+1}/a_n\|>1$	Inconclusive if lim $_{n\to\infty}$ $\|a_{n+1}/a_n\|=1$
Root Test	Σa_n, usually with $(a_n)^n$	Absolutely converges if lim $_{n\to\infty}$ $\|a_n\|^{1/n}<1$	Diverges if lim $_{n\to\infty}$ $\|a_n\|^{1/n}>1$	Inconclusive if lim $_{n\to\infty}$ $\|a_n\|^{1/n}=1$
Integral Test	a_n≡f(x), which is a positive, continuous, decreasing function on the interval [k,∞), usually with clearly-integrable functions	Converges if $\int_k^\infty$ f(x)dx converges	Diverges if $\int_k^\infty$ f(x)dx diverges	
Absolute/Conditional Convergence Classification	Σa_n	Absolutely converges if $\Sigma\|a_n\|$ converges	Diverges if Σa_n diverges	Conditionally converges if $\Sigma\|a_n\|$ diverges, but Σa_n converges

For the series listed, assume each series to be an infinite series starting at n=k: $\Sigma^\infty_{n=k}=\Sigma$ as previously defined

If a test is inconclusive, use another test

The symbol [⟺] represents the relationship "if and only if" (often abbreviated to "iff"), meaning both sides of the statement must be true at the same time, or false at the same time

Special Series

Series	Summation Form	First five terms	Radius & Interval of Convergence
Power Series centered at a	$\Sigma C_n(x-a)^n$	$C_0+C_1(x-a)+C_2(x-a)^2+C_3(x-a)^3+C_4(x-a)^4+...$	(a-R, a+R), [a-R, a+R), (a-R, a+R], or [a-R, a+R]
Taylor Series centered at a	$\Sigma f^{(n)}(a)(x-a)^n/n!$	$f(a)+f'(a)(x-a)+f''(a)(x-a)^2/2!+f'''(a)(x-a)^3/3!+f^{(4)}(a)(x-a)^4/4!+...$	$\|x-a\|<R$
Maclaurin Series (Taylor Series centered at 0)	$\Sigma f^{(n)}(0)(x-0)^n/n!=\Sigma f^{(n)}(0)x^n/n!$	$f(0)+f'(0)x+f''(0)x^2/2!+f'''(0)x^3/3!+f^{(4)}(0)x^4/4!+...$	$\|x\|<R$
1/(1-x)	Σx^n	$1+x+x^2+x^3+x^4+...$	R=1, I=(-1,1)
e^x	$\Sigma x^n/n!$	$1+x+x^2/2!+x^3/3!+x^4/4!+...$	R=∞, I=(-∞,∞)
ln(1+x)	$\Sigma(-1)^n x^{n+1}/(n+1)$	$x-x^2/2+x^3/3-x^4/4+x^5/5-...$	R=1, I=(-1,1]
arctan(x)	$\Sigma(-1)^n x^{2n+1}/(2n+1)$	$x-x^3/3+x^5/5-x^7/7+x^9/9-...$	R=1, I=[-1,1]

Special Series (cont)

sin(x)	$\Sigma(-1)^n x^{2n+1}/(2n+1)!$	$x-x^3/3!+x^5/5!-x^7/7!+x^9/9!-...$	R=∞, I=(-∞,∞)
cos(x)	$\Sigma(-1)^n x^{2n}/(2n)!$	$1-x^2/2!+x^4/4!-x^6/6!+x^8/8!-...$	R=∞, I=(-∞,∞)
$(1+x)^k$	$\Sigma(^k{}_n)x^n=\Sigma((k(k-1)(k-2)(k-3)...(k-n+1))/n!)x^n$	$1+kx+k(k-1)x^2/2!+k(k-1)(k-2)x^3/3!+k(k-1)(k-2)(k-3)x^4/4!+...$	R=1
Taylor's Inequality	$\lvert R_n(x)\rvert \le M\lvert x-a\rvert^{n+1}/(n+1)!$, where M is a constant such that $\lvert f^{(n+1)}(x)\rvert \le M$ for all $\lvert x-a\rvert \le d$		

For the series listed, assume each series to be an infinite series starting at n=0: $\Sigma^{\infty}{}_{n=0}=\Sigma$ as previously defined

The Radius of Convergence, R, is typically found by using the Ratio Test or Root Test

$(^k{}_n)$ is the "binomial coefficient" (read as "k choose n"). $(^k{}_n)$ = k!/(n!(k-n)!)

$f^{(n)}$ means "the nth derivative of the function f"

n!=n(n-1)!=n(n-1)(n-2)!=n(n-1)(n-2)(n-3)!=...

n! = n(n-1)(n-2)(n-3)...*3*2*1

0!=1, 1!=1

Areas of Functions

Between two functions	$\int_a^b$ ((top function)-(bottom function))dA
Enclosed by a polar function	$½\int_a^b f(\theta)^2 d\theta$
Between two polar functions	$½\int_a^b$ ((outer polar function)^2-(inner polar function)^2)dθ

Area enclosed by a polar function is with respect to the pole, which is the origin

Average value of a function on an interval: $f_{avg}=1/(b-a)\int_a^b f(x)dx$

Volumes of Solids of Revolution

Disk	$\pi\int_a^b$ (radius)^{2}dr
Washer	$\pi\int_a^b$ ((outer radius)^2 - (inner radius))^{2}dr
Cylindrical Shell	$2\pi\int_a^b$ (radius)(height)dr

Disk: slices are perpendicular and connected to axis of rotation
Washer: slices are perpendicular and disconnected to axis of rotation
Cylindrical Shells: slices are parallel to axis of rotation

Arc Lengths

Surface Areas

Function revolved about an axis	$2\pi\int_a^b$ (radius)(Arc Length component)ds
Function revolved about y-axis	$2\pi\int_a^b x\sqrt{(1+(f'(x))^2)}dx$
Function revolved about x-axis	$2\pi\int_a^b y\sqrt{(1+(g'(y))^2)}dy$
Parametric function of t revolved about y-axis	$2\pi\int_a^b f(x)\sqrt{((x'(t))^2+(y'(t))^2)}dt$
Parametric function of t revolved about x-axis	$2\pi\int_a^b g(y)\sqrt{((x'(t))^2+(y'(t))^2)}dt$

f'(x)=dy/dx, g'(y)=dx/dy, x'(t)=-dx/dt, and y'(t)=dy/dt

Integration by Parts

Indefinite Integral	$\int udv=uv-\int vdu$
Definite Integral	$\int_a^b udv=uv\vert_a^b - \int_a^b vdu$

Integration by Parts is used to integrate integrals that have components multiplied together in their simplest form, often referred to as a "product rule for integrals"

Choosing the "dv" term depends

Trigonometric Integrals

https://cheatography.com/crossant/cheat-sheets/integral-trigonometry/

Integration by Partial Fractions

(ax+b)/(px+q)(rx+s)	A/(px+q) + B/(rx+s)
$(ax+b)/(px+q)^2$	A/(px+q) + $B/(px+q)^2$
$(ax^2+bx+c)/((px+q)(rx^2+sx+t))$	A/(px+q) + (Bx+C)/(rx^2+sx+t)
$(ax^3+bx^2+cx+d)/(rx^2+sx+t)^2$	$(Ax+B)/(rx^2+sx+t)$ + $(Cx+D)/(rx^2+sx+t)^2$

For numerator/denominator N/D, partial fractions simplify integrals of polynomial rational expressions where deg(N) < deg(D), and a fully-factored D, into a sum of simpler fractions. If deg(N) ≥ deg(D), then polynomial long division must be used beforehand.

A non-repeated denominator factor of degree n gets a partial fraction with numerator degree n-1.

An irreducible denominator factor repeated p times (like (ax+b)^p) of degree n gets p partial fractions, all with numerator-degree n-1.

Improper Integrals (cont)

Divergence of $\int f(x)dx$	$\lim_{t\to\pm\infty} f(x)dx=\pm\infty$ or DNE

Improper Integrals are integrals with bounds at infinity (Type 1) or at least one discontinuity on the integrated region (Type 2)

Conic Sections

https://cheatography.com/crossant/cheat-sheets/conic-sections/

Parametric Curves and Polar Functions

Function	$\int_a^b \sqrt{(1+(f'(x))^2)}dx$
Parametric Function	$\int_a^b \sqrt{((x'(t))^2+(y'(t))^2)}dt$
Polar Function	$\int_a^b \sqrt{(r(\theta)^2+(r'(\theta))^2)}d\theta$

For standard functions: f'(x)=-dy/dx
For parametric functions: x'(t)=-dx/dt and y'(t)=dy/dt
For polar functions: r'(θ)=dr/dθ

on what will simplify the integral the best, while being relatively simple to integrate

The constant of integration does not need to be inserted until the integral has been fully simplified

Improper Integrals

$\int_a^{\infty} f(x)dx$	$\lim_{t\to\infty} \int_a^t f(x)dx$
$\int_{-\infty}^b f(x)dx$	$\lim_{t\to-\infty} \int_t^b f(x)dx$
$\int_{-\infty}^{\infty} f(x)dx$	$\lim_{t\to-\infty} \int_t^c f(x)dx + \lim_{t\to\infty} \int_x^t f(x)dx$
Convergence of $\int f(x)dx$	$\lim_{t\to\pm\infty}$ f(x)dx=L, where L is a constant

Parametric Curve C as a function of Parameter t	(x,y)=(f(t),g(t)) for t on [a,b]
Slope at a given point	dy/dx=-(dy/dt)/(dx/dt)
Second derivative	$d^2y/dx^2=(dy/dt)/(dx/dt)^2$
Polar Curve C as a function of Parameters r and θ	(r,θ)=(r,θ±2πn)=(-r,θ±πn)
Slope at a given point	dy/dx=-(dy/dθ)/(dx/dθ)
Cartesian/Rectangular to Polar coordinates	x=rcos(θ), y=rsin(θ)
Polar to Cartesian/Rectangular coordinates	$r^2=x^2+y^2$ or $r=\sqrt{(x^2+y^2)}$, tanθ=y/x or θ=arctan(y/x)

(dx/dt)≠0, (dx/dθ)≠0

Integral Approximations and Error Bounds

Midpoint Rule	$\Delta x(f(\bar{x}_1)+f(\bar{x}_2)+f(\bar{x}_3)+...+$ $_1)+f(\bar{x}_n))$

Integral Approximations and Error Bounds (cont)

Trapezoidal Rule	$(\Delta x/2)(f(x_1)+2f(x_2)+2f(x_3)+...+2f(x_{n-1})+f(x_1))$
Simpson's Rule	$(\Delta x/3)(f(x_1)+4f(x_2)+2f(x_3)+4f(x_4)+2f(x_5))+...+2f(x_{n-2})+4f(x_{n-1})+f(x_n))$
Midpoint Rule Error Bound	$\lvert E_m\rvert \le k(b-a)^3/24n^2$, $k=f''(x)_{max}$ on [a,b]
Trapezoidal Rule Error Bound	$\lvert E_t\rvert \le k(b-a)^3/12n^2$, $k=f''(x)_{max}$ on [a,b]
Simpson's Rule Error Bound	$\lvert E_s\rvert \le k(b-a)^5/180n^4$, $k=f^{(4)}(x)_{max}$ on [a,b]

Integral Approximations are typically used to evaluate an integral that is very difficult or impossible to integrate

$\Delta x=(b-a)/n$

$\bar{x}=(x_{i-1}+x_i)/2$, the average/median of two points x_{i-1} and x_i

Simpson's Rule can only be used if the given n is even, that is, n=2k for some integer k

In order of most accurate to least accurate approximation: Simpson's Rule, Midpoint Rule, Trapezoidal Rule, Left/Right endpoint approximation