Series

Series Type	General Summation	Convergence	Divergence	Notes
Infinite Series	$\Sigma a_n = \Sigma^{\infty}_{n=k} a_n$	Converges if lim $_{n\to\infty}$ S_n=L	Diverges if lim $_{n\to\infty}$ =∞ or DNE	S_n is the partial sum of the series: $S_n=a_1+a_2+a_3+a_4+...+a_{n-1}+a_n$
Harmonic Series	Σ1/n	Never converges	Always diverges	The alternating version of this series ($\Sigma(-1)^{n+1}/n$) converges, and Σ1/n is a P-Series with p=1
Geometric Series	$\Sigma^{\infty}_{n=0} ar^n = \Sigma^{\infty}_{n=1} ar^{n-1}$	Converges if |r|<1	Diverges if |r|≥1	If the series converges, its sum is S=a/(1-r)
P-Series	$\Sigma 1/n^p$	Converges if p>1	Diverges if p≤1	
Alternating Series	$\Sigma(-1)^n b_n$, $\Sigma(-1)^{n+1} b_n$, or $\Sigma(-1)^{n-1} b_n$	Converges if lim $_{n\to\infty}$ =0 and b_n is a decreasing sequence ($b_{n+1} \le b_n$ for all n)	Cannot show divergence, inconclusive	
Telescoping Series	$\Sigma(b_n - b_{n+1})$	Converges if lim $_{n\to\infty}$ =L	Diverges if lim $_{n\to\infty}$ S_n=∞ or DNE	S_n is the partial sum of the series: $S_n=\Sigma^n_{i=1}(b_i - b_{i+n})$ where n is finite

Alternating Series Estimation Theorem: If $S_n=\Sigma^n_{i=1}(-1)^n b_n$ or $\Sigma^n_{i=1}(-1)^{n-1} b_n$ is the sum of an alternating series that converges, then $|R_n|=|S-S_n| \le b_{n+1}$

Trigonometric functions like cos(nπ) or sin(nπ+π/2) act as sign alternators, like $(-1)^n$

The Alternating Series Test does not show divergence, however, implementing the test requires a Test For Divergence, which does show divergence

Series Tests

Test Type	Typical series to use test	Convergence	Divergence	Notes
Test for Divergence	Σa_n	Cannot show convergence, inconclusive	Diverges if lim $_{n\to\infty}$ ≠0	

Series Tests (cont)

Integral Test	Σa_n=f(n), which is a positive, continuous, decreasing function on the interval [k,∞), usually with easily-integrable functions	Converges if $\int_k^\infty$ f(n)dn converges	Diverges if $\int_k^\infty$ f(n)dn diverges	
(Direct) Comparison Test	a_n and b_n are positive-termed ($a_n \geq 0$ and $b_n \geq 0$ for all n) and $a_n \leq b_n$ for all n	Σa_n converges if Σb_n converges	Σb_n diverges if Σa_n diverges	Inconclusive if b_n diverges or a_n converges
Limit Comparison Test	a_n and b_n are positive-termed, and $\lim_{n\to\infty} a_n/b_n$=c, or $\lim_{n\to\infty} b_n/a_n$=d, where c and d are finite constants greater than 0	Σa_n converges ⟺ Σb_n converges	Σa_n diverges ⟺ Σb_n diverges	Inconclusive if either c or d=0 or ∞
Ratio Test	Σa_n, usually with n! terms or a_n^n	Absolutely converges if $\lim_{n\to\infty} \lvert a_{n+1}/a_n \rvert <1$	Diverges if $\lim_{n\to\infty} \lvert a_{n+1}/a_n \rvert >1$	Inconclusive if $\lim_{n\to\infty} \lvert a_{n+1}/a_n \rvert =1$
Root Test	Σa_n, usually with a_n^n	Absolutely converges if $\lim_{n\to\infty} \lvert a_n \rvert^{1/n}<1$	Diverges if $\lim_{n\to\infty} \lvert a_n \rvert^{1/n}>1$	Inconclusive if $\lim_{n\to\infty} \lvert a_n \rvert^{1/n}=1$
Absolute/-Conditional Convergence	Σa_n	Absolutely converges if $\Sigma \lvert a_n \rvert$ converges	Conditionally converges if $\Sigma \lvert a_n \rvert$ diverges, but Σa_n converges	Diverges if Σa_n diverges

For the series listed, assume each series to be an infinite series starting at n=k: $\Sigma^\infty_{n=k}=\Sigma$

If Test for Divergence passes ($\lim_{n\to\infty}$ =0), use another test

The symbol [⟺] represents the relationship "if and only if" (often abbreviated to "iff"), meaning both sides of the statement must be true at the same time, or false at the same time

If a test is inconclusive, use another test

Special Series

Series	Summation Form	First five terms	Radius and Interval of Convergence
Power Series centered at a	$\Sigma C_n(x-a)^n$	$C_0+C_1(x-a)+C_2(x-a)^2+C_3(x-a)^3+C_4(x-a)^4+...$	

Special Series (cont)

Taylor Series centered at a	$\Sigma f^{(n)}(a)(x-a)^n/n!$	$f(a)+f'(a)(x-a)+f''(a)(x-a)^2/2!+f'''(a)(x-a)^3/3!+f^{(4)}(a)(x-a)^4/4!+...$	R>|x-a|
Maclaurin Series (Taylor Series centered at 0)	$\Sigma f^{(n)}(0)(x-0)^n/n!=\Sigma f^{(n)}(0)x^n/n!$	$f(0)+f'(0)x+f''(0)x^2/2!+f'''(0)x^3/3!+f^{(4)}(0)x^4/4!+...$	
1/(1-x)	Σx^n	$1+x+x^2+x^3+x^4+...$	R=1, I=(-1,1)
e^x	$\Sigma x^n/n!$	$1+x+x^2/2!+x^3/3!+x^4/4!+...$	R=∞, I=(-∞,∞)
ln(1+x)	$\Sigma(-1)^{n+1}x^n/n$	$x-x^2/2+x^3/3-x^4/4+x^5/5+...$	R=1, I=(-1,1]
arctan(x)	$\Sigma(-1)^n x^{2n+1}/(2n+1)$	$x-x^3/3+x^5/5-x^7/7+x^9/9+...$	R=1, I=[-1,1]
sin(x)	$\Sigma(-1)^n x^{2n+1}/(2n+1)!$	$x-x^3/3!+x^5/5!-x^7/7!+x^9/9!+...$	R=∞, I=(-∞,∞)
cos(x)	$\Sigma(-1)^n x^{2n}/(2n)!$	$1-x^2/2!+x^4/4!-x^6/6!+x^8/8!+...$	R=∞, I=(-∞,∞)
$(1+x)^k$	$\Sigma({}^k{}_n)x^n=\Sigma((k(k-1)(k-2)(k-3)...(k-n+1))/n!)x^n$	$1+kx+k(k-1)x^2/2!+k(k-1)(k-2)x^3/3!+k(k-1)(k-2)(k-3)x^4/4!+...$	R=1
Taylor's Inequality	$\|R_n(x)\|\leq M\|x-a\|^{n+1}/(n+1)!$, given $M\geq\|f^{(n+1)}(x)\|$ for all $\|x-a\|\leq d$		

For the series listed, assume each series to be an infinite series starting at n=0: $\Sigma^{\infty}_{n=0}=\Sigma$

Note that the formula for a Degree 1 Taylor Polynomial, $T_1(x)$, has the same formula as the Linear Approximation formula learned in Calculus I

$f^{(n)}$ means "the nth derivative of the function f"

n!=n(n-1)!=n(n-1)(n-2)!=n(n-1)(n-2)(n-3)!=...

n! = n(n-1)(n-2)(n-3)...*3*2*1

0!=1, 1!=1

Areas of Functions

Between two functions	$\int_a^b$ ((top function)-(bottom function))dA
Enclosed by a polar function	$\frac{1}{2}\int_a^b f(\theta)^2 d\theta$
Between two polar functions	$\frac{1}{2}\int_a^b$ ((outer polar function)^2-(inner polar function)^2)dθ

Area enclosed by a polar function is with respect to the pole, which is the origin

Average value of a function: $f_{avg}=1/(b-a)\int_a^b f(x)dx$

Volumes of Solids of Revolution

Disk	$\pi\int_a^b (\text{radius})^2 dV$
Washer	$\pi\int_a^b (\text{outer radius})^2 - (\text{inner radius})^2 dV$
Cylindrical Shell	$2\pi\int_a^b$ (radius)(height)dV

For Cylindrical Shells: radius=x or y, and height=f(x) or g(y)

Arc Lengths

Function	$\int_a^b \sqrt{(1+(f'(x))^2}dx$
Parametric Function	$\int_a^b \sqrt{((x'(t))^2+(y'(t))^2}dt$
Polar Function	$\int_a^b \sqrt{(r(\theta)^2+(r'(\theta))^2}d\theta$

For standard functions: f'(x)=-dy/dx
For parametric functions: x'(t)=-dx/dt and y'(t)=dy/dt
For polar functions: r'(θ)=dr/dθ

Surface Areas

Function revolved about an axis	$2\pi\int_a^b$ (radius)(Arc Length component)ds
Function revolved about y-axis	$2\pi\int_a^b x\sqrt{(1+(f'(x))^2}dx$
Function revolved about x-axis	$2\pi\int_a^b y\sqrt{(1+(g'(y))^2}dy$
Parametric function of t revolved about y-axis	$2\pi\int_a^b f(x)\sqrt{((x'(t))^2+(y'(t))^2}dt$
Parametric function of t revolved about x-axis	$2\pi\int_a^b g(y)\sqrt{((x'(t))^2+(y'(t))^2}dt$

f'(x)=dy/dx, g'(y)=dx/dy, x'(t)=-dx/dt, and y'(t)=dy/dt

Integration by Parts

Indefinite Integral	$\int udv=uv-\int vdu$
Definite Integral	$\int_a^b udv=uv\|_a^b - \int_a^b vdu$

Integration by Parts is used to integrate integrals that have components multiplied together in their simplest form, often referred to as a "product rule for integrals"

Choosing the "dv" term depends on what will simplify the integral the best, while being relatively simple to integrate

The constant of integration does not need to be inserted until the integral has been fully simplified

Trigonometric Integrals

https://cheatography.com/crossant/cheat-sheets/integral-trigonometry/

Integration by Partial Fractions

$(px+q)/((x-a)(x-b))$	$A/(x-a) + B/(x-b)$
$(px+q)/(x-a)^2$	$A/(x-a) + B/(x-a)^2$
$(px^2+qx+r)/(-(x-a)(x-b)(x-c))$	$A/(x-a) + B/(x-b) + C/(x-c)$
$(px^2+qx+r)/((x-a)^2(x-b))$	$A/(x-a) + B/(x-a)^2 + C/(x-b)$
$(px^2+qx+r)/((-x-a)(x^2+bx+c))$	$A/(x-a) + Bx+C/(x^2+bx+c)$
$\int 1/(a^2+x^2)\, dx$	$(1/a)\arctan(-x/a)+C$

Integration by Partial Fractions is used to simplify integrals of polynomial rational expressions into simpler fractions with a factored, irreducible denominator

The degree (highest power) of the numerator's polynomial must be less than the degree of the denominator's polynomial, otherwise, polynomial long division must be used before converting the expression into partial fractions

Improper Integrals

$\int_a^\infty f(x)dx$	$\lim_{t\to\infty} \int_a^t f(x)dx$
$\int_{-\infty}^b f(x)dx$	$\lim_{t\to-\infty} \int_t^b f(x)dx$
$\int_{-\infty}^\infty f(x)dx$	$\lim_{t\to-\infty} \int_t^c f(x)dx + \lim_{t\to\infty} \int_x^t f(x)dx$

Improper Integrals (cont)

Convergence of $\int f(x)dx$	$\lim_{t\to\pm\infty} =L$
Divergence of $\int f(x)dx$	$\lim_{t\to\pm\infty} =\pm\infty$ or DNE

Improper Integrals are integrals with bounds at infinity (Type 1) or discontinuous bounds (Type 2)

Conic Sections

https://cheatography.com/crossant/cheat-sheets/conic-sections/

Parametric Curves and Polar Functions

Parametric Curve C as a function of Parameter t	(x,y)=(f(t),g(t)) for t on [a,b]
Slope at a given point	dy/dx=-(dy/dt)/(dx/dt)
Second derivative	$d^2y/dx^2=(dy/dt)/(dx/dt)^2$
Polar Curve C as a function of Parameter θ	(r,θ)=(r,θ±2πn)=(-r,θ±πn)
Slope at a given point	dy/dx=-(dy/dθ)/(dx/dθ)
Cartesian/Rectangular to Polar coordinates	x=rcos(θ), y=rsin(θ)
Polar to Cartesian/Rectangular coordinates	$r^2=x^2+y^2$ or $r=\sqrt{(x^2+y^2)}$, tanθ=y/x or θ=arctan(y/x)

(dx/dt)≠0, (dx/dθ)≠0

Integral Approximations and Error Bounds

Midpoint Rule	$\Delta x(f(\bar{x}_1)+f(\bar{x}_2)+f(\bar{x}_3)+...+f(\bar{x}_{n-1})+f(\bar{x}_n))$
Trapezoidal Rule	$(\Delta x/2)(f(x_1)+2f(x_2)+2f(x_3)+...+2f(x_{n-1})+f(x_1))$
Simpson's Rule	$(\Delta x/3)(f(x_1)+4f(x_2)+2f(x_3)+4f(x_4)+2f(x_5))+...+2f(x_{n-2})+4f(x_{n-1})+f(x_n))$
Midpoint Rule Error Bound	$\lvert E_m\rvert \leq k(b-a)^3/24n^2$, $k=f''(x)_{max}$ on [a,b]
Trapezoidal Rule Error Bound	$\lvert E_t\rvert \leq k(b-a)^3/12n^2$, $k=f''(x)_{max}$ on [a,b]
Simpson's Rule Error Bound	$\lvert E_s\rvert \leq k(b-a)^5/180n^4$, $k=f^{(4)}(x)_{max}$ on [a,b]

Integral Approximations are typically used to evaluate an integral that is very difficult or impossible to integrate

$\Delta x=(b-a)/n$

$\bar{x}=(x_{i-1}+x_i)/2$, the average/median of two points x_{i-1} and x_i

Simpson's Rule can only be used if the given n is even, that is, n=2k for some integer k.

In order of most accurate to least accurate approximation: Simpson's Rule, Midpoint Rule, Trapezoidal Rule, Left/Right endpoint approximation