Rules of Differentiation for a fn. of 1 Variable

Constant Function Rule

y = f(*x*) = k

d/dy(f(*x*)) = d/dy(k) = 0

Power Rule

y = f(*x*) = x^n

d/dy(f(*x*)) = d/dy(x^n) = n.x^{n-1}

Power Function Rule Generalized

y = f(*x*) = cx^n (where c is the multiplicative constant

d/dy(f(*x*)) = d/dy(cx^n) = cnx^{n-1}

Rules of Diff. for 2 or > fn. of the same variable

Sum & Diff Rule

d/dy [f(*x*) ± g(*x*)] = d/dy [f(*x*)] ± d/dy [g(*x*)] = f '(x) ± g'(x)

Product Rule

d/dy [f(*x*).g(*x*)] = f(*x*).d/dy [g(*x*)] + g(*x*).d/dy [f(*x*)] = f(*x*).g'(x) + g(*x*).f '(x)

Finding Marginal Revenue Function from Average Revenue Function

AR = 15 - Q, TR = AR.Q = 15Q-Q^2, MR = 15 - 2Q

AR = TR/Q = P.Q/Q = P , therefore [Average Revenue is the inverse of the demand as AR is the curve relating Price to output: P = f(Q)] Under Perfect Competition, the AR curve is horizontal straight line (MR - AR =), therefore they must coincide). Under imperfect competition, AR is a downward sloping curve (where MR - AR < 0 for all positive levels of output, MR is above AR)

Quotient Rule

d/dy [f(*x*)/g(*x*)] = f '(x).g(*x*) + g'(x).f(*x*)/$[g(x)]^2$

Relationship Between Marginal Cost & Average Cost Functions

AC = C(Q)/Q as long as Q>0, The rate of change of AC WRT Q is done by differentiating AC.

AC' = 1/Q [C'(Q) - C(Q)/Q], MC = C'(Q)

Exercise 1:

A firm produces ‘q’ units of water. Its total revenue and total cost functions are given as:

TR=-0.5q^2+100q and TC=5q^2+12q+200

a) Calculate MR and interpret your solution.

MR = dTR/dQ = -q + 100 [It is a decreasing function, therefore a downward sloping curve, MR is the extra revenue that an additional unit of q will bring to the firm, when q = 0, MR is 100 implying that an additional unit of q will decrease the extra revenue]

b) Calculate MC and interpret your solution.

MC = dTC/dQ = 10q + 12 [It is an increasing function, therefore a upward sloping curve, MC is the extra cost that an additional unit of q will bring to the firm, when q = 0, MC = 12 implying that an additional unit of q will increase the extra cost]

c) How many units of water should the firm produce to maximize total profit?

Rules of Diff. for 2 or > fn. of the same variable (cont)

Profit Max = MR = MC

-q+100=10q+12

11q=88

q=8

Partial Differentiation

Partial Derivatives

f 1 = ∂y/∂*x*1, this is the partial derivative WRT *x*1, all other independent variables in function are held constant

Techniques of Partial Differentiation

1.*Simple Rule*

y = f(X1,X2) = 3$X^2$1 + X1X2 + 4$X^2$2

∂y/∂x1 = 6X1+X2; ∂y/∂x2 = X1+8X2

2. *Product Rule*

y = f(u,v) = (u+4)(3u+2v)

fu = (u+4)(3) + (3u+2v)(1) = 6u + 2v + 12

fv = (u+4)(2) + (3u+2v)(0) = 2u+8

3.*Quotient Rule*

y = f(u,v) = (3u-2v)/(u^2 + 3v)

fu = [(3)(u^2 + 3v) - (2u)(3u-2v)]/$[(u^2 + 3v)]^2$ *simplify it*

fv = [(-2)(u^2 + 3v) - (3)(3u-2v)]/$[(u^2 + 3v)]^2$ *simplify it*

A partial derivative is a measure of the instantaneous rates of change of some variable.

MPPk = Marginal physical Product of Capital, which is the partial derivative Qk relates to the changes of output WRT infinitesimal changes in capital while labour input is held constant. Similar the partial Derivative of Ql is the mathematical representation of the MPPL function.

Exercise 4:

Given the following utility function: f(x,y,z)=x^2 y^3 z 0<x,y,z<1

a) Calculate marginal utility with respect to y and interpret.

∂U/∂y = 3x^2y^2z [MU is + & > 0 for all values of x,y,z, function is strictly monotonic]

Rules of Diff. with fns. of Different Variables

Chain Rule

z = f(y), y = g(x); dz/dx = (dz/dy).(dy/dx) = f '(y). g'(x)

function can extend to 3 variables:

z = f(y), y = g(x), x = h(w)

dz/dw = (dz/dy).(dy/dx).(dx/dw) = f '(y). g'(x). h '(w)

Rules of Diff. with fns. of Different Variables (cont)

Given that TR = f(Q), where Q is a function of labour input, Q = g(L), find dTR/dL, by chain rule, we get dTR/dL = (dTR/dQ).(dQ/dL) = f '(Q).g'(L), [dTR/dQ is the MR function and dQ/dL is the marginal-physical-product of Labour (MPPL) function. dTR/dL has the connotation of the Marginal revenue product of labour (MRPL)

Exercise 2:

Given TR= PQ and $Q=A.K^{0.5}L^{0.3}H^{0.2}$

a) Calculate ∂TR/∂K. Interpret your solution.

$\partial TR/\partial K = (dTR/dQ).(dQ/dK) = P.[A.0.5.K^{-0.5}.L^{0.3}H^{0.2}]$

$= 0.5.P.A.L^{0.3}H^{0.2}/K^{0.5}$, this is the Marginal Revenue Product of Capital, how does total revenue change when an extra unit of output is produced when one unit of capital is added, dQ/dK is the marginal product of capital (MPK).

b) Calculate ∂Q/∂H. Interpret your solution.

$\partial Q/\partial H = 0.2\ A.K^{0.5}L^{0.3}/H^{0.8}$, this is the marginal product of Human Capital, it is the extra amount of output that is produced when one unit of Human capital is added, holding all the other inputs constant.

Inverse Function Rule

If a function y=f(x) represents a one-to-one mapping (for every x value, it has a unique y value, vice versa) then f has an inverse f^{-1}.

A function is strictly increasing if:

X1>X2 --> f(X1) > f(X2) for all values of x, y

A function is strictly decreasing if:

X1>X2 --> f(X1) < f(X2) for all values of x, y

If a function is either strictly increasing or strictly decreasing it is strictly monotonic.

If a function is monotonic then the function has an inverse.

Thus for example, firms demand curve Q = f(P) that has a negative slope throughout is decreasing, as such it has an inverse function, P = f^{-1} (Q) which gives the Av. revenue of the firm P = AR.

For inverse functions, the rule of differentiation: dx/dy = 1/[dy/dx]

check for strictly monotonic: f′(x) > 0 or f′(x) <0 for all x. If function is a U-shaped curve, may have to check which part of the curve is increasing and which part is decreasing for the x values.

Exercise 3:

Using the TR and TC functions from Exercise 1:

a) Determine whether the total cost function is strictly increasing?

$TC=5q^2+12q+200$, if q>0 then TC will increase

MC = 10q+12, it will be positive for all values of q if q>0

b) For what values of q will the total revenue function be strictly increasing?

$TR=-0.5q^2+100q$

MR=-q+100 (when q = 0 , MR = 100 which is the turning point for TR) therfore TR will be increasing for q values that are < 100

Application to Comparative Static Analysis

Market Model

Look at page 170 - 172 in textbook

National - Income Model

Look at page 172 - 173 in textbook, it goes over the income equilibrium and the partial derivative to get the government expenditure multiplier, non-income tax multiplier and how an increase in the income tax rate will lower equilibrium income.

Note on Jacobian Determinants

Consider the two functions

$$\begin{aligned} y_1 &= 2x_1 + 3x_2 \\ y_2 &= 4x_1^2 + 12x_1x_2 + 9x_2^2 \end{aligned} \qquad (7.24)$$

If we get all the four partial derivatives

$$\frac{\partial y_1}{\partial x_1} = 2 \qquad \frac{\partial y_1}{\partial x_2} = 3 \qquad \frac{\partial y_2}{\partial x_1} = 8x_1 + 12x_2 \qquad \frac{\partial y_2}{\partial x_2} = 12x_1 + 18x_2$$

and arrange them into a square matrix in a prescribed order, called a Jacobian matrix and denoted by J, and then take its determinant, the result will be what is known as a *Jacobian determinant* (or a *Jacobian*, for short), denoted by $|J|$:

$$|J| \equiv \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ (8x_1 + 12x_2) & (12x_1 + 18x_2) \end{vmatrix} \qquad (7.25)$$

For economy of space, this Jacobian is sometimes also expressed as

$$|J| \equiv \left| \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} \right|$$

The two equations are linearly dependent if |J| = 0