final

Final Exam 1

1. Prove or disprove that if A and B are sets then $A \cap (A \cup B) = A$.

2. Find the prime factorization of 16575.

3. (a) Prove or disprove: If $a \equiv b \pmod 5$, where a and b are integers, then $a^2 \equiv b^2 \pmod 5$.

(b) Prove or disprove: If $a^2 \equiv b^2 \pmod 5$, where a and b are integers, then $a \equiv b \pmod 5$.

4. Use mathematical induction to prove that $n! \geq 2^{n-1}$ whenever n is a positive integer.

5. Suppose that $a_1 = 10$, $a_2 = 5$, and $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 3$. Prove that 5 divides a_n whenever n is a positive integer.

6. How many bit string of length 10 have at least one 0 in them?

7. (a) How many functions are there from a set with three elements to a set with four elements?

(b) How many are one-to-one?

(c) How many are onto?

8. A door lock is opened by pushing a sequence of buttons. Each of the three terms in the combination is entered by pushing either one button or two buttons simultaneously. If there are 5 buttons, how many different combinations are there? (Example: 1-3, 2, 2-4 is a valid combination.)

9. Find a recurrence relation and initial conditions for the number of ways to go up a flight of stairs if stairs can be climbed one, two, or three at a time.

0. How many positive integers not exceeding 1000 are not divisible by either 8 or 12?

1. (a) Show that the relation $R = \{ (x,y) \mid x - y \text{ is an even integer} \}$ is an equivalence relation on the set of real numbers.

(b) What are the equivalence classes of 1 and $\frac{1}{2}$ with respect to R?

2. Answer the following questions about the graph $K_{3,4}$.

(a) How many vertices and how many edges are in this graph?

(b) Is this graph planar? Justify your answer.

(c) Does this graph have an Euler circuit? Does it have an Euler path? Give reasons for your answers.

(d) What is the chromatic number of this graph?

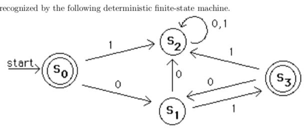
3. Find a spanning tree for the graph $K_{3,4}$ using

(a) a depth-first search.

(b) a breadth-first search.

4. Find the sum-of-products expansion of the Boolean function $f(x,y,z)$ that has the value 1 if and only if an odd number of the variables x, y, and z have the value 1.

5. Find the set recognized by the following deterministic finite-state machine.



6. A fair coin is flipped until a tail first appears, at which time no more flips are made.

(a) What is the probability that exactly five flips are made?

(b) What is the expected number of flips?

final2

Final Exam 2

1. Prove or disprove that $\overline{(A-B)} = \overline{A} \cup B$ whenever A and B are sets.

2. Prove or disprove that the fourth power of an odd positive integer always leaves a remainder of 1 when divided by 16.

3. Use mathematical induction to prove that every postage of greater than 5 cents can be formed from 3-cent and 4-cent stamps.

4. How many bit strings of length 10 have at least eight 1's in them?

5. (a) How many functions are there from a set with four elements to a set with three elements?

(b) How many of these functions are one-to-one?

(c) How many are onto?

6. How many symmetric relations are there on a set with eight elements?

7. (a) Let m be a positive integer greater than 2. Show that the relation R consisting of those ordered pairs of integers (a,b) with $a \equiv \pm b \pmod m$ is an equivalence relation.

(b) Describe the equivalence classes of this relation where $m = 4$.

8. (a) Does the graph $K_{2,5}$ have an Euler circuit? If not, does it have an Euler path?

(b) Does the graph $K_{2,5}$ have a Hamilton path?

9. How many nonisomorphic unrooted trees are there with four vertices? Draw these trees.

10. Construct a binary search tree from the words of the sentence *This is your discrete mathematics final*, using alphabetical order, inserting words in the order they appear in the sentence.

11. Find the sum-of-products expansion for the Boolean function $x + y + z$.

12. (a) Describe the bit strings that are in the regular set represented by $\mathbf{0^*11(0 \cup 1)^*}$?

(b) Construct a nondeterministic finite-state automaton that recognizes this set.

13. A thumb tack is tossed until it first lands with its point down, at which time no more tosses are made. On each toss, the probability of the tack's landing point down is 1/3.

(a) What is the probability that exactly five tosses are made?

(b) What is the expected number of tosses?

final 1 a

Final Exam 1 Solutions

1. Suppose that $x \in A$. Then $x \in A \cup B$, so $x \in A \cap (A \cup B)$. Conversely, suppose that $x \in A \cap (A \cup B)$. Then $x \in A$. Hence $A \cap (A \cup B) = A$.

2. We find that 2 does not divide 16575, but 3 does with $16575/3 = 5525$. We see that 3 does not divide 5525, but 5 does with $5525/5 = 1105$. We see that 5 divides 1105 with $1105/5 = 221$. We see that neither 5, 7, nor 11 divides 221. However, 13 does, with $221/13 = 17$. Hence $16575 = 3 \cdot 5^2 \cdot 13 \cdot 17$.

3. (a) Suppose that $a \equiv b \pmod 5$. Then $5 \mid a - b$, so there is an integer k such that $a - b = 5k$. It follows that $a^2 - b^2 = (a+b)(a-b) = (a+b)5k$. Hence $a^2 - b^2 = 5l$, where $l = (a+b)k$. It follows that $5 \mid a^2 - b^2$, so $a^2 \equiv b^2 \pmod 5$.

(b) We see that $1^2 \equiv 4^2 \pmod 5$, but $1 \not\equiv 4 \pmod 5$.

4. The basis step follows since $1! = 1 = 2^0$. For the inductive hypothesis assume that $n! \geq 2^{n-1}$. Then $(n+1)! = (n+1) \cdot n! \geq (n+1) \cdot 2^{n-1} \geq 2 \cdot 2^{n-1} = 2^n$.

5. The basis step is completed by noting that $a_1 = 10$ and $a_2 = 5$ are both divisible by 5. For the inductive step assume that a_k is divisible by 5 for every positive integer k with $k < n$, where $n \geq 3$. It follows that $a_n = 2a_{n-1} + 3a_{n-2}$ is divisible by 5, since the sum of two integers divisible by 5 is also divisible by 5.

6. The number of bit strings of length 10 with at least one 0 in them is the number of all bits strings of length 10 minus the number of bits strings of length 10 with no 0's in them. This is $2^{10} - 1 = 1024 - 1 = 1023$.

7. (a) There are $4^3 = 64$ functions from a set with three elements to a set with four elements.

(b) There are $4 \cdot 3 \cdot 2 = 24$ one-to-one functions from a set with three elements to a set with four elements.

(c) There are no onto functions from a set with three elements to a set with four elements.

8. A push of buttons in the combinations is either the push of one of the five buttons or the simultaneous push of one of $C(5,2) = 10$ combinations of two of the five buttons. Hence there are $15 \cdot 15 \cdot 15 = 3375$ possible combinations for the door lock.

9. Let a_n denote the number of ways to climb n stairs if stairs can be climbed one, two, or three at a time. Suppose that n is a positive integer, $n \geq 4$. A person can climb n stairs by going up $n-1$ stairs and then climbing one stair, by going up $n-2$ stairs and then climbing two stairs, or by going up $n-3$ stairs and then climbing three stairs. Hence $a_n = a_{n-1} + a_{n-2} + a_{n-3}$. Note that $a_1 = 1$ since there is only one way to climb one stair, $a_2 = 2$ since two stairs can be climbed one stair at a time or two stairs at once, and $a_3 = 4$ since we can climb three stairs by taking stairs one at a time, by going up two stairs followed by one stair, by going up one stair followed by two stairs, or by taking all three stairs at once.

10. The number of integers not exceeding 1000 that are not divisible by either 8 or 12 is 1000 minus the number of these integers that are divisible by either 8 or 12. Using the principle of inclusion–exclusion, we see that there are $\lfloor 1000/8 \rfloor + \lfloor 1000/12 \rfloor - \lfloor 1000/24 \rfloor = 125 + 83 - 41 = 167$ such integers, where we used the fact that the integers divisible by both 8 and 12 are those divisible by $\mathrm{lcm}(8,12) = 24$. Hence there are $1000 - 167 = 833$ positive integers not exceeding 1000 that are not divisible by either 8 or 12.

11. (a) Since $x - x = 0$ is an even integer for every real number x it follows that R is reflexive. If $(x,y) \in R$ then $x - y$ is an even integer. It follows that $y - x = -(x-y)$ is also an even integer. Hence R is symmetric. Now suppose that $(x,y) \in R$ and $(y,z) \in R$. Then $x - y$ and $y - z$ are even integers. Since $x - z = (x-y) + (y-z)$ and the sum of two even integers is again even, it follows that $x - z$ is also an even integer. This shows that R is transitive. We conclude that R is an equivalence relation.

(b) We have $[1]_R = \{ x \mid x - 1 \text{ is an even integer} \}$. Hence $[1]_R = \{ x \mid x = 1 + 2k \text{ where } k \text{ is an integer} \}$. In other words, $[1]_R$ is the set of odd integers. Similarly, $[\frac{1}{2}]_R = \{ x \mid x - \frac{1}{2} \text{ is an even integer} \}$. Hence $[\frac{1}{2}]_R = \{ x \mid x = \frac{1}{2} + 2k \text{ where } k \text{ is an integer} \}$. This is the set $\{\ldots, -\frac{7}{2}, -\frac{3}{2}, \frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \ldots\}$.

12. (a) The graph $K_{3,4}$ has $3 + 4 = 7$ vertices and $3 \cdot 4 = 12$ edges.

(b) $K_{3,4}$ is not planar since it contains a subgraph isomorphic to $K_{3,3}$, which is not planar.

(c) $K_{3,4}$ has three vertices of degree 4 and four vertices of degree 3. Since there are more than two vertices of odd degree, there is no Euler path in this graph, and therefore also no Euler circuit.

final 2a

Final Exam 2 Solutions

1. This set equality can be proved in several different ways. One method is to use set identities already known to hold. We find that $\overline{(A-B)} = \overline{(A \cap \overline{B})} = \overline{A} \cup \overline{(\overline{B})} = \overline{A} \cup B$, where we have used De Morgan's laws and the double complementation law.

2. Suppose that a is an odd integer. Then $a = 2k+1$. We have $a^4 = (2k+1)^4 = 16k^4 + 32k^3 + 24k^2 + 8k + 1 = 16(k^4 + 2k^3) + 8k(3k+1) + 1$. If k is even then $8k = 16l$ where $k = 2l$, so $a^4 = 16N + 1$, where N is an integer. If k is odd then $3k + 1 = 3(2l+1) + 1 = 6l + 4 = 2m$, where $m = 3l + 2$, so again $a^4 = 16N + 1$, where N is an integer. It follows that $a \equiv 1 \pmod{16}$ whenever a is an odd integer.

3. The basis step is completed by noting that postage of 6 cents can be formed using two 3-cent stamps. Now assume that postage of n cents can be formed, where n is a positive integer greater than or equal to 6. If a 3-cent stamp was used to form n cents postage, replace this stamp with a 4-cent stamp to obtain postage of $n+1$ cents. Otherwise, if only 4-cent stamps were used, then at least two of them were used, so replace two 4-cent stamps with three 3-cent stamps to obtain postage of $n+1$ cents.

4. The number of bit strings of length 10 with at least eight 1's in them equals the number with exactly eight 1's plus the number with exactly nine 1's plus the number with exactly ten 1's. There are $C(10,8) = 10!/(2!8!) = 45$ such strings with exactly eight 1's, $C(10,9) = 10$ such strings with exactly nine 1's, and $C(10,10) = 1$ such string with exactly ten 1's. Hence there are $45 + 10 + 1 = 56$ bit strings of length 10 containing at least eight 1's.

5. (a) There are $3^4 = 81$ functions from a set with four elements to a set with three elements.

(b) There are no one-to-one functions from a set with four elements to a set with three elements since $4 > 3$.

(c) There are $3^4 - C(3,2)2^4 + C(3,1)1^4 = 81 - 48 + 3 = 36$ onto functions from a set with four elements to a set with three elements.

6. A symmetric relation is determined by specifying whether (i,j) and (j,i) belong to this relation for the pairs with $i \neq j$, and whether (i,i) belongs to the relation for all elements i in the set. Since there are eight elements in the set, there are $C(8,2) = 28$ pairs (i,j) and (j,i) with $i \neq j$, and eight elements i. Hence there are $2^{28+8} = 2^{36}$ symmetric relations on a set with eight elements.

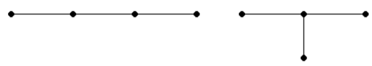
7. (a) Let a be an integer. Then $a \equiv a \pmod m$ since $m \mid a - a$. It follows that R is reflexive. Now suppose that $(a,b) \in R$. Then $a \equiv b \pmod m$ or $a \equiv -b \pmod m$. It is easy to see that $b \equiv a \pmod m$ or $b \equiv -a \pmod m$. Hence $(b,a) \in R$. It follows that R is symmetric. Now assume that $(a,b) \in R$ and $(b,c) \in R$. Then $a \equiv b \pmod m$ or $a \equiv -b \pmod m$, and $b \equiv c \pmod m$ or $b \equiv -c \pmod m$. We can easily see that each of the four combinations leads to $a \equiv c \pmod m$ or $a \equiv -c \pmod m$. Hence $(a,c) \in R$, and R is transitive.

(b) Let $m = 4$. The equivalence classes of R are $[0]_R = \{ a \in \mathbf{Z} \mid a \equiv 0 \pmod 4 \} = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$, $[1]_R = \{ a \in \mathbf{Z} \mid a \equiv \pm 1 \pmod 4 \} = \{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$, and $[2]_R = \{ a \in \mathbf{Z} \mid a \equiv 2 \pmod 4 \} = \{\ldots, -6, -2, 2, 6, \ldots\}$.

8. (a) The graph $K_{2,5}$ has two vertices of degree 5 and five vertices of degree 2. Hence it has an Euler path and no Euler circuit.

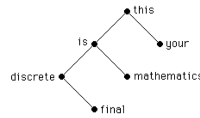
(b) There is no Hamilton path in this graph since any path containing all five vertices of degree 2 must visit some of the vertices of degree 5 more than once.

9. There are two nonisomorphic unrooted trees with four vertices, as shown.



final2

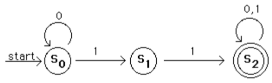
10. We construct the following binary search tree.



11. The Boolean function $x + y + z$ has the value 1 unless $x = y = z = 0$, so it has the value 1 for the other seven combinations of the values of these variable. Hence the sum-of-products expansion is $f(x,y,z) = xyz + xy\overline{z} + x\overline{y}z + x\overline{y}\,\overline{z} + \overline{x}yz + \overline{x}y\overline{z} + \overline{x}\,\overline{y}z$.

12. (a) The strings in this set are those that begin with an arbitrary number of 0's followed by two consecutive 1's, followed by an arbitrary bit string.

(b) The following nondeterministic finite-state automaton recognizes this set.



13. The number of tosses follows a geometric distribution with parameter $p = 1/3$.

(a) The tack must land point up four times in a row and then point down, and the probability of this is $(2/3)^4(1/3) = 16/243$.

(b) The expected number of tosses in a geometric distribution is $1/p = 3$.

final 1a1

(d) The chromatic number of $K_{3,4}$ is 2 since this graph is bipartite.

13. A depth-first spanning tree of $K_{3,4}$ is shown on the left and a breadth-first spanning tree of $K_{3,4}$ is shown on the right. (Note that the first answer depends on the part of the graph you start in.)

14. The sum-of-products expansion is $f(x,y,z) = xyz + x\overline{y}\,\overline{z} + \overline{x}y\overline{z} + \overline{x}\,\overline{y}z$.

15. The set recognized is the set represented by $(\mathbf{01})^*$.

16. The number of flips follows a geometric distribution with parameter $p = 1/2$.

(a) The coin must land heads four times in a row and then tails, and the probability of this is $(1/2)^5 = 1/32$.

(b) The expected number of flips in a geometric distribution is $1/p = 2$.